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Shock Profile Solutions of the Boltzmann Equation*

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Abstract. Shock waves in gas dynamics can be described by the Euler Navier–Stokes, or Boltzmann equations. We prove the existence of shock profile solutions of the Boltzmann equation for shocks which are weak. The shock is written as a truncated expansion in powers of the shock strength, the first two terms of which come exactly from the Taylor tanh(x) profile for the Navier–Stokes solution. The full solution is found by a projection method like the Lyapunov–Schmidt method as a bifurcation from the constant state in which the bifurcation parameter is the difference between the speed of sound c_0 and the shock speed s.

1. Introduction

Shock waves are one of the most important features of gas dynamics. They can be understood from several different theories, and for steady plane shock waves the different descriptions have been well developed mathematically. By the Euler equations, and the resulting Rankine–Hugoniot conditions, a shock is described as a jump discontinuity in density, velocity, and temperature from $(\rho_-, \mathbf{u}_-, T_-)$ on the left to $(\rho_+, \mathbf{u}_+, T_+)$ on the right, which translates steadily at speed s[4]. If viscosity and heat conduction are included through the compressible Navier–Stokes equations, the shock wave is found to be a smooth profile which translates uniformly at speed s and smoothly interpolates between the asymptotic values $(\rho_-, \mathbf{u}_-, T_-)$ at $x = -\infty$ and $(\rho_+, \mathbf{u}_+, T_+)$ at $x = +\infty$ [9, 21].

For a weak shock this provides shock profiles very close to those observed experimentally. But for strong shock waves more realistic results are obtained from the Boltzmann equation of kinetic theory, which includes a statistical description of the molecular interactions within the gas. The Boltzmann shock profile translates uniformly at speed s and interpolates between two velocity distribution functions $F_-(\xi)$ at $x = -\infty$ and $F_+(\xi)$ at $x = +\infty$ which are uniform

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Maxwellians given by

$$F_{\pm}(\xi) = \rho_{\pm}(2\pi T_{\pm})^{3/2} \exp\{-|\boldsymbol{\xi} - \mathbf{u}_{\pm}|^2/2T_{\pm}\}, \tag{1.1}$$

and in which $(\rho_-, \mathbf{u}_-, T_-)$, $(\rho_+, \mathbf{u}_+, T_+)$, and s satisfy the Rankine–Hugoniot conditions. The distributions F_+ and F_- are independent of $\mathbf{x}$ and t and are equilibrium solutions of the Boltzmann equation. The resulting profiles, determined either numerically [19] or by analytic approximation [13, 20], agree very well with experiments. The excellent review article by Fiszdon, Herczynski, and Walenta [7] contains detailed comparisons of the Navier–Stokes and Boltzmann solutions with experimental results.

In this paper we prove the existence of shock profile solutions of the Boltzmann equation for weak shocks and demonstrate the agreement of these solutions with the Navier–Stokes profiles for such shocks. The solution is found as a truncated expansion in powers of the shock strength. The first term is the uniform Maxwellian state; the next has spatial variation given by the $\tanh(x)$ profile of a weak Navier–Stokes shock. The higher order terms approach constant values at $x = \pm\infty$, but at the rate $e^{-\varepsilon|x|} + e^{-|x|^\beta}$ with $0 < \beta \leq 1$, which depends on the intermolecular force law. By contrast the tanh profile decays like $e^{-\varepsilon|x|}$.

The intermolecular forces considered here are those which derive from hard cut-off potentials as defined by Grad [11]. They are related to power law forces $\mathscr{F}(r) = r^{-s}$; the decay exponent is then given by $\beta = 2(3-\gamma)^{-1}$ with $\gamma = \dfrac{s-5}{s-1}$ Nicolaenko and Thurber [15] already proved an analogous result for the hard sphere potential with $s = \infty$ and $\beta = 1$. The slower decay rate $\beta < 1$ for other potentials was previously indicated by several authors [17, 24, 25]. It is caused by the long mean free paths of molecules of high velocity. Their collision frequency is given by the function $\nu(\xi) \approx (1 + |\xi|)^\gamma$ (cf. (2.8)) and their mean free path by $\xi\nu(\xi)^{-1}$. For $s < \infty, \gamma < 1$ and the mean free path $\uparrow \infty$ as $|\xi| \uparrow \infty$. Thus fact particles travel a long distance before equilibrating, i.e. before becoming part of the Maxwellian distributions at $x = \pm\infty$. This slow equilibration is balanced against the small number of large velocity particles in the distribution (1.1) to obtain the overall decay rate $e^{-|x|^\beta}$. There is a similar phenomenon in the initial value problem for soft potentials investigated by Caflisch [2] and Ukai and Asano [22].

Shock profiles for a model Boltzmann equation with discrete velocities were constructed by Gatignol [8] and Caflisch [1]. The agreement between the Boltzmann equation and the Euler or Navier–Stokes equations away from shocks was shown by Nishida [16], Kawashima, Matsumura, and Nishida [12], and Caflisch [3]. The projection method used here is compared with the Chapman–Enskog expansion in [23].

The nonlinear Boltzmann equation is described in Sect. 2 and Appendix A and is specialized to Eq. (2.13) and (2.14) for the steady plane shock profile. The main result on the existence of shock profiles is stated in Theorem 2.1. The equations are analyzed by a projection method, like the Lyapunov–Schmidt method, in Sect. 3 to find the weak shock profile as a bifurcation from the constant state, in which the perturbation parameter ε is the difference between the sound speed c_0 and the shock speed s. This is the same as the method introduced in [14] and

[15] by Nicolaenko and Thurber. In this problem we are unable to find an exact eigenfunction for the projection method; instead an approximate eigenfunction is used. After a partial expansion of the solution and a modification to eliminate the null space in the second Lyapunov–Schmidt equation, the equations are written as (3.52)–(3.55). The first two are solved explicitly; the third is a simple near-linear scalar equation.

The analysis of Eq. (3.55) occupies Sect. 4, 5, and 6. Basic estimates on the linear collision operator are derived in Sect. 4. These use new estimates on the collision kernel and a new result, Proposition 4.4, showing compactness in the sup norm for the collision operator (a more limited result was proved by Grad [11]). In Sect. 5 these estimates are used to construct a semi-group to solve the linearized equation. Decay of the linearized solution is demonstrated in Sect. 6. Using this decay, the full nonlinear equations are solved in Sect. 7.

We use italics for a vector $\xi \in R^3$ and non-italics for its magnitude $\xi = |\xi|$. We also write ξ_1 for the first component of ξ.

2. The Boltzmann Equation for a Shock Profile

The nonlinear Boltzmann equation of kinetic theory is

$$\left(\frac{\partial}{\partial t} + \xi \cdot \frac{\partial}{\partial \mathbf{x}}\right) F = Q(F, F), \tag{2.1}$$

in which $F = F(\xi, \mathbf{x}, t)$ is the distribution function for gas particles with velocity $\xi \in R^3$ at position $\mathbf{x} \in R^3$ and time $t \in R^+$. The collision operator Q is a quadratic integral operator over ξ and is described in detail in Appendix A. In the collision process mass, momentum, and energy are conserved, i.e. for any distributions F and G

$$\begin{aligned} &\langle 1, Q(F,G)\rangle = 0, \\ &\langle \xi_i, Q(F,G)\rangle = 0, \qquad i = 1,2,3, \\ &\langle \xi^2, Q(F,G)\rangle = 0, \end{aligned} \tag{2.2}$$

in which $\langle f, g\rangle = \int_{R^3} f(\xi) g(\xi) d\xi$. The local equilibrium distributions for the scattering are distributions F with $Q(F,F) = 0$; the only solutions are the Maxwellians

$$F(\xi) = \rho (2\pi T)^{-3/2} \exp\{-(\xi - \mathbf{u})^2/2T\}.$$

Since x and t are mere parameters in Q, the constants $\rho, \mathbf{u}, T$ may depend arbitrarily on $\mathbf{x}$ and t. For any distribution F, symmetry and positivity properties of F imply that

$$\int \log(F(\xi)) Q(F,F)(\xi) d\xi < 0. \tag{2.3}$$

A plane steady shock profile is a continuous solution $F(\xi, x, t) = F(\xi, x - st)$ which depends on only one space variable $x = x_1$ and translates at uniform speed s. Its values at $x = \pm\infty$ are Maxwellians $F_\pm$ given by (1.1) with $\rho_\pm, u_\pm, T_\pm$ each constant. By shifting and rescaling ξ, F, and s we can replace them by

$$F(\xi, -\infty) = \omega_-(\xi) \equiv (2\pi)^{-3/2} e^{-\xi^2/2}, \tag{2.4}$$
$$F(\xi, \infty) = \omega_+(\xi) \equiv \rho_+(2\pi T_+)^{3/2} \exp\{-((\xi_1 - u_+)^2 + \xi_2^2 + \xi_3^2)/2T_+\},$$

and ask that F solve

$$(\xi_1 - s)\frac{\partial}{\partial x} F = Q(F, F). \tag{2.5}$$

Next we linearize F about ω_- by setting $F = \omega_- + \omega_-^{1/2} f$ so that f solves

$$(\xi_1 - s)\frac{\partial}{\partial x} f = -Lf + \nu\Gamma(f, f), \tag{2.6}$$
$$f(\xi, -\infty) = 0,$$
$$f(\xi, -\infty) = f_\infty(\xi) = (\omega_+ - \omega_-)\omega_-^{-1/2}. \tag{2.7}$$

The operators $Lf = -2\omega_-^{-1/2} Q(\omega_-, \omega_-^{1/2} f)$ and $\nu\Gamma(f, g) = \omega_-^{-1/2} Q(\omega_-^{1/2} f, \omega_-^{1/2} g)$ and the function $\nu(\xi)$ are described in detail in Appendix A and in [10]. Several important properties are that

$$\begin{aligned} Lf(\xi) &= \nu(\xi) f(\xi) - Kf(\xi), \\ Kf(\xi) &= \int_{R^3} k(\xi, \eta) f(\eta) d\eta, \\ \nu_1(1+\xi)^\gamma &< \nu(\xi) < \nu_2(1+\xi)^\gamma, \end{aligned} \tag{2.8}$$

with $0 \leqq \gamma \leqq 1$, $0 < \nu_1$, and $0 < \nu_2$ each constant. The function $\nu(\xi)$ is locally uniformly continuous and the operator L is self-adjoint and non-negative with $N(L) = R(L)^\perp$ spanned by the orthonormal sequence $\{\chi_0, \ldots, \chi_4\}$ defined by

$$\begin{aligned} \chi_0 &= \omega_-^{1/2}, \\ \chi_i &= \xi_i \omega_-^{1/2}, \\ \chi_4 &= 6^{-1/2}(\xi^2 - 3)\omega_-^{1/2}. \end{aligned} \tag{2.9}$$

The operator K is compact in $L^2(\xi)$.

The spatially uniform distributions $f = 0$ and $f = f_\infty$ are both solutions of (2.6). The desired continuous solution connecting these two states must satisfy the following conservation properties, which come from (2.2) and (2.6):

$$\langle \chi_i(\xi_1 - s), f(\xi, x) \rangle = 0, \qquad \text{for all } x, \tag{2.10}$$

and for $i = 0, \ldots, 4$. For $x = \infty$, these are just the Rankine–Hugoniot jump conditions for the states (ρ_+, u_+, T_+) and $(1, 0, 1)$ and the speed s, viz.

$$\begin{aligned} -s(\rho_+ - 1) + \rho_+ u_+ &= 0, \\ -s\rho_+ u_+ + \rho_+ u_+^2 + \rho_+ T_+ - 1 &= 0, \\ -s\{\rho_+(\tfrac{3}{2}T_+ + \tfrac{1}{2}u_+^2) - \tfrac{3}{2}\} + \rho_+ u_+(\tfrac{3}{2}T_+ + \tfrac{1}{2}u_+^2) + \rho_+ u_+ T_+ &= 0. \end{aligned} \tag{2.11}$$

Note however that the Rankine–Hugoniot condition (2.10) holds for all x.

From (2.3) it follows that $\frac{\partial}{\partial x}\int(\xi_1-s)F\log F\,d\xi<0$ and in particular $\int(\xi_1-s)\omega_-\log\omega_-\,d\xi>\int(\xi_1-s)\omega_+\log\omega_+\,d\xi$. This is the analogue of the Boltzmann H-Theorem for the shock problem. The integrals are calculated using (2.4) to obtain the inequality

$$s\tfrac{3}{2}(\log 2\pi+1)>\rho_+(u_+-s)(-\log\rho_++\tfrac{3}{2}\log 2\pi T_+-\tfrac{3}{2}).$$

The entropy function for an ideal monotonic gas as considered here is $S=\frac{3}{2}\cdot\log(\frac{3}{2}\rho^{-2/3}T)$. So this inequality can be rewritten using (2.11) as $s(S_--S_+)>0$, the entropy inequality across a shock in which S_+ and S_- are the entropies of the fluid states (ρ_+,u_+,T_+) and $(1,0,1)$ at $x=\pm\infty$. This is equivalent to the usual entropy condition

$$s(1-\rho_+)>0. \tag{2.12}$$

The relations (2.11) and (2.12) are conditions on the choice of ω_+. We take $s\geqq 0$; then if $s\geqq c_0=(5/3)^{1/2}$, the sound speed of an ideal monatomic gas, the only choice is $\omega_+=\omega_-$ and the solution of (2.6), (2.7) is $f\equiv 0$. If $0<s<c_0$, there is a solution $\omega_+\neq\omega_-$ (cf. [4]).

We shall study only weak shocks with $c_0-s=\varepsilon>0$ small; (2.11) then implies that $\omega_+-\omega_-=0(\varepsilon)$ (cf. [4]). We shall also find that the spatial variation of f is at the rate ε. Thus we replace x, f, and f_∞ in (2.6) and (2.7) by $x'=\varepsilon x, f'=\varepsilon^{-1}f, f'_\infty=\varepsilon^{-1}f_\infty$. Dropping the primes, the equations are rewritten as

$$(\xi_1-s)\frac{\partial}{\partial x}f=-\frac{1}{\varepsilon}Lf+\nu\Gamma(f,f), \tag{2.13}$$

$$f(\xi,-\infty)=0,$$

$$f(\xi,\infty)=f_\infty(\xi)=\varepsilon^{-1}(\omega_+-\omega_-)\omega_-^{-1/2}. \tag{2.14}$$

The solution f of (2.13), (2.14) will be compared with the solution of the Navier–Stokes (NS) equations, which in the original unscaled variables are

$$-s\frac{\partial}{\partial x}\rho+\frac{\partial}{\partial x}(\rho u)=0,$$

$$-s\frac{\partial}{\partial x}\rho u+\frac{\partial}{\partial x}(\rho u^2+p)=\tfrac{4}{3}\eta\frac{\partial^2}{\partial x^2}u, \tag{2.15}$$

$$-s\frac{\partial}{\partial x}\rho(e+\tfrac{1}{2}u^2)+\frac{\partial}{\partial x}\{\rho u(e+\tfrac{1}{2}u^2)+pu\}=\frac{2}{3}\frac{\partial}{\partial x}\lambda\frac{\partial}{\partial x}e+\tfrac{4}{3}\eta\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right),$$

$$p=\rho T=\tfrac{2}{3}\rho e. \tag{2.16}$$

The viscosity and heat conduction coefficients η and λ are determined by the first term F_1 in the Chapman–Enskog expansion [6] as

$$\tfrac{4}{3}\eta\frac{\partial}{\partial x}u=-\langle\xi_1\omega_-^{1/2},(\xi_1-s)F_1\rangle. \tag{2.17}$$

$$\tfrac{2}{3}\lambda\frac{\partial}{\partial x}e+\tfrac{4}{3}\eta u\frac{\partial u}{\partial x}=-\langle\tfrac{1}{2}\xi^2\omega_-^{1/2},(\xi_1-s)F_1\rangle. \tag{2.18}$$

For a weak shock $s=c_0-\varepsilon$ and $\frac{\partial e}{\partial x}=(c_0^{-1}+o(\varepsilon))\frac{\partial u}{\partial x}$ and the equations (2.15) can be combined to yield approximately

$$\tfrac{4}{3}c_0u^2-2\varepsilon c_0u=\tfrac{4}{3}c_0(\eta+\tfrac{1}{5}\lambda)u_x, \tag{2.19}$$

with the right hand side given by

$$\langle c_0\xi_1+\tfrac{1}{3}\xi^2,(\xi_1-s)F_1\rangle. \tag{2.20}$$

The solution of (2.18), after rescaling as above, is

$$u_{\mathrm{NS}}=\tfrac{3}{4}(\tanh(-\tfrac{3}{4}(\eta+\tfrac{1}{5}\lambda)^{-1}x)+1). \tag{2.21}$$

Denote the corresponding density and temperature profiles by ρ_{NS} and T_{NS} and define

$$\begin{aligned}f_{\mathrm{NS}}(\xi,x)&=\omega_-^{-1/2}(\xi)\rho_{\mathrm{NS}}(2\pi T_{\mathrm{NS}})^{-3/2}\\&\quad\cdot\exp\{-((\xi_1-u_{\mathrm{NS}})^2+\xi_2^2+\xi_3^2)/2T_{\mathrm{NS}}\}.\end{aligned} \tag{2.22}$$

The results will be proved using weighted sup norms on ξ defined by

$$\begin{aligned}\|f\|_{\alpha,r}&=\sup(1+\xi)^r e^{\alpha\xi^2}|f(\xi)|,\\ \|f\|_r&=\|f\|_{0,r},\end{aligned} \tag{2.23}$$

and function spaces

$$\begin{aligned}G_{\alpha,r}&=\{f:\|f\|_{\alpha,r}<\infty\},\\ G_r&=G_{0,r}.\end{aligned} \tag{2.24}$$

Decay in x will be measured by the function

$$A(x)=e^{-\mu|x/\varepsilon|^\beta}+e^{-\tau_1|x|}, \tag{2.25}$$

in which μ,β, and τ_1 will be chosen later.

Theorem 2.1. *Let s,ρ_+,u_+,T_+ satisfy conditions (2.11) and (2.12) with $\varepsilon=c_0-s>0$ sufficiently small. Let f_{NS} be the distribution defined by (2.22). Then there is a shock profile solution f of the Boltzmann equation (2.13) and (2.14). It satisfies:*

$$\begin{aligned}\|f(x)-f_{\mathrm{NS}}(x)\|_r&\leqq c\varepsilon A(x),\\ \|f(x)-f_{\mathrm{NS}}\|_{\alpha,r}&\leqq c\varepsilon.\end{aligned} \tag{2.26}$$

Moreover f is unique, up to translation in x, among those solutions satisfying (2.26).

It can actually be shown that f is unique, up to translation, among those solutions which are bounded in $G_{\alpha,r}$. This means that F is unique among solutions of the form $F=\omega_-+O(\varepsilon)$.

3. The Projection Method

We shall solve (2.13), (2.14) by a projection method similar to the Lyapunov–Schmidt method, in which the principal part of f is found as an eigenfunction ϕ_ε of the linearized problem (2.13), and the bifurcation parameter is $\varepsilon = c_0 - s$. Decompose f as $f(x, \xi) = z(x)\phi_\varepsilon(\xi) + \varepsilon w(x, \xi)$. The equation for z ((3.32) or (3.52) and (3.54)) will be fully nonlinear but easily solvable since z is a function only of x. The equation for w ((3.31) or (3.53) and (3.55)) will be weakly nonlinear since it makes only a small contribution to f. The function ϕ_ε is chosen to have the following properties:

(i) $f(x=\infty, \xi) - f(x=-\infty, \xi) = z_\infty \phi_\varepsilon(\xi) + O(\varepsilon^2)$, for some constant z_∞, so that ϕ_ε contains the dominant variation of f.

(ii) $L\phi_\varepsilon = \varepsilon\tau(\xi_1 - s)\phi_\varepsilon$, so that ϕ_ε is a generalized eigenfunction for the linear operator L in which the eigenvalue $\varepsilon\tau$ can be thought of as the Laplace transform variable for x.

(iii) ϕ_ε satisfies the constraints (2.10).

This method was used by Nicolaenko and Thurber [15] in their study of a shock in a gas composed of rigid spheres and further developed by Nicolaenko [14]. A similar eigenvalue problem was solved in [5, 26]. For other intermolecular force laws, we are unable to solve the eigenvalue problem exactly. The difficulty is that the (generalized) eigenvalue $\varepsilon\tau$ is embedded in the (generalized) continuous spectrum. Since $L = v(\xi) + K$ with K compact, an easy extension of Weyl's theorem [18] implies that the (generalized) continuous spectrum for the problem in (ii) is the set $\{\tau : v(\xi) = \varepsilon\tau(\xi_1 - s)\}$ which is the whole real line if $v(\xi)$ satisfies (2.8) with $\gamma < 1$. However it is sufficient in the projection method to use an approximate eigenfunction ϕ_ε solving

$$L\phi_\varepsilon = \varepsilon\tau_0(\xi_1 - s)\phi_\varepsilon + \varepsilon^2\mu_\varepsilon. \tag{3.1}$$

A. *The Approximate Eigenfunction*

We shall find ϕ_ε as a sum of the form

$$\phi_\varepsilon = \phi_0 + \varepsilon\psi_\varepsilon, \tag{3.2}$$

$$\psi_\varepsilon = \phi_1 + \varepsilon\theta_\varepsilon. \tag{3.3}$$

with ϕ_0 and ϕ_1 independent of ε and satisfying

$$L\phi_0 = 0, \tag{3.4}$$

$$L\phi_1 = \tau(\xi_1 - c_0)\phi_0, \tag{3.5}$$

$$\theta_\varepsilon \text{ bounded}, \tag{3.6}$$

$$\langle(\xi_1 - s)\chi_i, \phi_\varepsilon\rangle = 0, \quad i = 0, \ldots, 4, \tag{3.7}$$

$$\langle(\xi_1 - s)\phi_\varepsilon, \phi_\varepsilon\rangle = -\varepsilon. \tag{3.8}$$

By including more terms in the expansion of ϕ_ε we could make the error $\varepsilon^2\mu_\varepsilon$ as small as desired, but we are unable to show that the resulting series converges to an eigenfunction.

Proposition 3.1. *Let $\varepsilon = c_0 - s > 0$ be sufficiently small. Then there are $\phi_\varepsilon \in G_{\frac{1}{4}-3}$, $\mu_\varepsilon \in G_{\frac{1}{4}-3}$, and $\tau > 0$ which solve (3.2)–(3.8) with*

$$\|\mu_\varepsilon\|_{\frac{1}{4}-3} \leqq c \quad \text{independent of } \varepsilon, \tag{3.9}$$

$$\langle \chi_i, \mu_\varepsilon \rangle = 0. \tag{3.10}$$

Proof. From (3.4) and (3.5) it follows that $\phi_0 = \sum_{i=0}^{4} \alpha_i \chi_i$ and $\langle \chi_i, (\xi_1 - c_0)\phi_0 \rangle = 0$. The solution, constructed in Appendix B, is

$$\phi_0 = \bar{\alpha}\phi_0' = \bar{\alpha}(\chi_0 + c_0\chi_1 + (2/3)^{1/2}\chi_4), \tag{3.11}$$

with $c_0 = (5/3)^{1/2}$ and $\bar{\alpha}$ an undetermined scalar. Let $\phi_1 = \bar{\alpha}\left(\phi_1' + \sum_{i=0}^{4} \beta_i \chi_i\right)$ in which

$$L\phi_1' = \tau(\xi_1 - c_0)\phi_0', \tag{3.12}$$

$$\langle \chi_i, \phi_1' \rangle = 0, \quad i = 0, \ldots, 4. \tag{3.13}$$

This determines ϕ_1' uniquely. The scalars $\tau, \bar{\alpha}$, and β_i, and the function θ_ε are now found by the constraints (3.7) and (3.8), which can be written as

$$\langle \chi_i, \phi_0' \rangle + \left\langle (\xi_1 - c_0)\chi_i, \phi_1' + \sum_j \beta_j \chi_j \right\rangle = 0, \quad i = 0, \ldots, 3, \tag{3.14}$$

$$\langle \phi_0', \phi_0' \rangle + \langle (\xi_1 - c_0)\phi_0', \phi_1' \rangle = 0, \tag{3.15}$$

$$\langle \phi_0, \phi_0 \rangle + 2\langle (\xi_1 - c_0)\phi_0, \phi_1 \rangle = -1, \tag{3.16}$$

$$\langle \chi_i, \phi_1 \rangle + \langle (\xi_1 - s)\chi_i, \theta_\varepsilon \rangle = 0, \quad i = 0, \ldots, 4, \tag{3.17}$$

$$2\langle \phi_0, \phi_1 \rangle + \langle (\xi_1 - s)\phi_1, \phi_1 \rangle + \langle (\xi_1 - s)(2\phi_0 + 2\varepsilon\phi_1 + \varepsilon^2\theta_\varepsilon), \theta_\varepsilon \rangle = 0. \tag{3.18}$$

First we can rewrite (3.15) and (3.16) as

$$\tau = -\langle L\phi_1', \phi_1' \rangle / \langle \phi_0', \phi_0' \rangle < 0, \tag{3.19}$$

$$\bar{\alpha}^2 = -\tau \langle L\phi_1', \phi_1' \rangle^{-1} = 3/10. \tag{3.20}$$

The remaining equations can be solved for β_i and θ_ε as shown in Appendix B, with β_i independent of ε and θ_ε bounded independently of ε. From (3.15) and (3.16) the following useful relation is derived (which is not needed in this proof):

$$\langle \phi_0, \phi_0 \rangle = -\langle (\xi_1 - c_0)\phi_0, \phi_1 \rangle = 1. \tag{3.21}$$

The error term is $\mu_\varepsilon = \tau\phi_0 - \tau(\xi_1 - s)\psi_\varepsilon + L\theta_\varepsilon$ which satisfies (3.9) and (3.10) due to (3.7) and (3.5).

B. The Projection-Operators and the Lyapunov–Schmidt Equations.

Next we define projections

$$\begin{aligned} Pf &= -(\xi_1 - s)\phi_\varepsilon \langle \psi_\varepsilon, f \rangle, \\ \Pi f &= -\varepsilon^{-1}\phi_\varepsilon \langle (\xi_1 - s)\phi_\varepsilon, f \rangle, \end{aligned} \tag{3.22}$$

with the following properties, which are consequences of (2.9) and (3.1)–(3.10):

(i) $$\Pi^2 = \Pi, \quad P^2 = P; \tag{3.23}$$

(ii) If $\langle(\xi_1 - s)\chi_i, f\rangle = 0, \quad 0 \leqq i \leqq 4,$

$$P(\xi_1 - s)f = (\xi_1 - s)\Pi f; \tag{3.24}$$

(iii) If $\langle(\xi_1 - s)\chi_i, f\rangle = 0, \quad 0 \leqq i \leqq 4$, or $\langle \chi_i, g\rangle = 0, 0 \leqq i \leqq 4$,

$$\langle f, Pg\rangle = \langle \Pi f, g\rangle, \tag{3.25}$$

(iv) $$L_1 = (I - P)L(I - \Pi) \text{ is self-adjoint,}$$

(v) $$(I - P)Lf = L_1 + \varepsilon h_1,$$

$$PLf = \varepsilon\tau(\xi_1 - s)\Pi f + \varepsilon h_2, \tag{3.26}$$

$$h_1 = -\langle(\xi_1 - s)\phi_\varepsilon, f\rangle(I - P)\mu_\varepsilon = \varepsilon z(I - P)\mu_\varepsilon,$$

$$h_2 = -(\xi_1 - s)\phi_\varepsilon\langle \mu_\varepsilon, f\rangle. \tag{3.27}$$

In other words Π and P are adjoints of each other for functions satisfying (2.10), for such functions P passes through $(\xi_1 - s)$ to become Π, and P nearly passes through L with errors h_1 and h_2. We have replaced ϕ_ε by ψ_ε in P to eliminate the factor ε^{-1}. If $\langle \phi_0, f\rangle = 0$, this does not really change P.

Decompose f as

$$f(\xi, x) = z(x)\phi_\varepsilon(\xi) + \varepsilon w(\xi, x), \tag{3.28}$$

with

$$z\phi = \Pi f, \qquad w = \varepsilon^{-1}(I - \Pi)f. \tag{3.29}$$

The Lyapunov–Schmidt equations are found by multiplying (2.13) once by P and once by $\varepsilon^{-1}(I - P)$ and using (3.24) and (3.26) to obtain

$$(\xi_1 - s)\frac{\partial}{\partial x}z\phi_\varepsilon = -\tau(\xi_1 - s)z\phi_\varepsilon - h_2 + Pv\Gamma(f, f), \tag{3.30}$$

$$(\xi_1 - s)\frac{\partial}{\partial x}w = -\varepsilon^{-1}L_1 w - \varepsilon^{-1}h_1 + \varepsilon^{-1}(I - P)v\Gamma(f, f). \tag{3.31}$$

If (3.30), is divided by $(\xi_1 - s)\phi_\varepsilon$ we find

$$\frac{\partial}{\partial x}z = -\tau z + \gamma z^2 + \varepsilon h_3, \tag{3.32}$$

$$\gamma = -\langle \psi_\varepsilon, v\Gamma(\phi_\varepsilon, \phi_\varepsilon)\rangle,$$

$$h_3 = \langle \mu_\varepsilon, z\psi_\varepsilon + w\rangle - \langle \psi_\varepsilon, v\Gamma(2\phi_\varepsilon z + \varepsilon w, w). \tag{3.33}$$

C. Removal of the Null Space.

Next we modify (3.31) to remove the null space of L_1. Define two more projections

$$\begin{aligned} P_0 f &= -(\xi_1 - c_0)\phi_0\langle \phi_1, f\rangle, \\ \Pi_0 f &= -\phi_1\langle(\xi_1 - c_0)\phi_0, f\rangle, \end{aligned} \tag{3.34}$$

which are independent of ε, and denote

$$L_2 f = (I - P_0) L (I - \Pi_0) f. \tag{3.35}$$

Then $P - P_0 = O(\varepsilon)$ and $L(\Pi - \Pi_0) = O(\varepsilon)$ so that

$$L_1 f = L_2 f + \varepsilon L_3 f, \tag{3.36}$$

with

$$L_3 f = \varepsilon^{-1}\{-(P - P_0) Lf - L(\Pi - \Pi_0) f + (PL\Pi - P_0 L \Pi_0) f\}, \tag{3.37}$$

and L_3 is bounded. For convenience in notation define

$$\chi_{-1} = \phi_1. \tag{3.38}$$

Proposition 3.2. *$N(L_2)$ is spanned by $\{\chi_i, i = -1, \ldots, 4\}$.*

Proof. For any f and h, $\langle f, P_0 h\rangle = \langle \Pi_0 f, h\rangle$ and thus $\langle f, L_2 f\rangle = \langle (I - \Pi_0) f, L(I - \Pi_0) f\rangle$. Since $L \geqq 0$ then $L_2 f = 0$ if and only if $L(I - \Pi_0) f = 0$ which means that f solves $Lf = cL\phi_1$ with $c = -\langle (\xi_1 - c_0)\phi_0, f\rangle$. Other than multiplication by a factor and addition of $\chi_i (i = 0, \ldots, 4)$, this can have at most one solution; thus $\dim N(L_2) \leqq 6$. On the other hand $L\chi_i = \Pi_0 \chi_i = 0$ so that $\chi_i \in N(L_2)$ for $i = 0, \ldots, 4$. Also by (3.21), $L\phi_1 = cL\phi_1$, and $\phi_1 \in N(L_2)$, which concludes the proof.

Now define

$$\begin{aligned}
K_1 f &= \langle (\xi_1 - s)\psi_\varepsilon, f\rangle (\xi_1 - s)\psi_\varepsilon + \sum_{i=0}^{4} \langle (\xi_1 - s)\chi_i, f\rangle (\xi_1 - s)\chi_i,\\
K_2 f &= \sum_{i=-1}^{4} \langle (\xi_1 - c_0)\chi_i, f\rangle (\xi_1 - c_0)\chi_i,\\
K_3 &= \varepsilon^{-1}(K_1 - K_2),\\
M &= L_2 + K_2,\\
M_3 &= L_3 + K_3.
\end{aligned} \tag{3.39}$$

The operators K_3 and M_3 are bounded. As in (2.8) the operator M, which is independent of ε, can be represented as

$$\begin{aligned}
Mf &= v(\xi) f + Hf,\\
Hf(\xi) &= \int \bar{k}(\xi, \eta) f(\eta) d\eta.\\
\bar{k}(\xi, \eta) &= -k(\xi, \eta) + \sum_{i=-1}^{4} (\xi_1 - c_0)\chi_i(\xi)(\eta_1 - c_0)\chi_i(\eta)\\
&\quad + (\xi_1 - c_0)\phi_0(\xi) L\phi_1(\eta) + L\phi_1(\xi)(\eta_1 - c_0)\phi_0(\eta)\\
&\quad + (\xi_1 - c_0)\phi_0(\xi)\langle \phi_1, L\phi_1\rangle (\eta_1 - c_0)\phi_0(\eta).
\end{aligned} \tag{3.40}$$

The equation (3.31) will be replaced by the following equation:

$$(\xi_1 - s)\frac{\partial}{\partial x} w = -\frac{1}{\varepsilon} Mw - M_3 w + \varepsilon^{-1} h_4, \tag{3.41}$$

$$h_4 = \varepsilon z (I - P)\mu_\varepsilon + (I - P) v \Gamma(f, f). \tag{3.42}$$

This is motivated and justified by the next proposition.

Proposition 3.3. (i) *M is self-adjoint and strictly positive*; (ii) *If w solves (3.41) and $\langle(\xi_1 - s)\psi_\varepsilon, w\rangle = \langle(\xi_1 - s)\chi_i, w\rangle = 0$, $i = 0,\ldots,4$ at $x = \pm\infty$, then w solves (3.31).*

Proof. Since L_2 and K_2 are self-adjoint, so is M. First note that $K_2 \geqq 0$ and $L_2 \geqq 0$. In Appendix B, it is shown that $\det\{\langle(\xi_1 - c_0)\chi_i, \chi_j\rangle, i = -1,\ldots,4, j = -1,\ldots,4\} \neq 0$. Now

$$\left\langle \sum_{i=-1}^{4} \alpha_i\chi_i, K_2 \sum_{i=-1}^{4} \alpha_i\chi_i \right\rangle = \sum_{i=-1}^{4} \left(\sum_{j=-1}^{4} \alpha_j \langle(\xi_1 - c_0)\chi_i, \chi_j\rangle \right)^2 > 0, \tag{3.43}$$

since at least one of the squared terms must be nonzero. Thus K_2 is strictly positive on $\{\sum\alpha_i\chi_i\} = N(L_2)$ and the combination $M = L_2 + K_2$ is strictly positive.

To demonstrate (ii) we first rewrite the right hand side of (3.41) as $-\varepsilon^{-1}(L_1 + K_1)w + \varepsilon^{-1}h_4$. For any g and h,

$$\langle\psi_\varepsilon, (I - P)h\rangle = \langle\chi_i, Ph\rangle = \langle\chi_i, \nu\Gamma(g,h)\rangle = \langle\chi_i, \mu_\varepsilon\rangle = 0, \qquad (i = 0,\ldots,4).$$

Therefore the inner product of (3.41) with ψ_ε and χ_i results in

$$\begin{aligned}
\frac{\partial}{\partial x}\langle\xi_1 - s)\psi_\varepsilon, w\rangle &= -\varepsilon^{-1}\langle\psi_\varepsilon, K_1 w\rangle \\
&= -\varepsilon^{-1}\left\{a\langle(\xi_1 - s)\psi_\varepsilon, w\rangle + \sum_{j=0}^{4} a_j\langle(\xi_1 - s)\chi_j, w\rangle\right\}, \\
\frac{\partial}{\partial x}\langle(\xi_1 - s)\chi_i, w\rangle &= -\varepsilon^{-1}\langle\chi_i, K_1 w\rangle \\
&= -\varepsilon^{-1}\left\{b_i\langle(\xi_1 - s)\chi_i, w\rangle + \sum_{j=0}^{4} b_{ij}\langle(\xi_1 - s)\chi_j, w\rangle\right\},
\end{aligned} \tag{3.44}$$

with a, a_j, b, b_{ij} constants. The boundary conditions at $x = \pm\infty$ in (ii) insure that $\langle(\xi_1 - s)\chi_i, w\rangle = \langle(\xi_1 - s)\psi_\varepsilon, w\rangle = 0$, so that $K_1 w = 0$ and (3.41) becomes exactly (3.31).

D. Elimination of the Asymptotic Values

From (2.11), (2.14), and (3.11), it follows that

$$f_\infty = \varepsilon^{-1}\bar{\alpha}^{-1}(\rho_+ - 1)\phi_\varepsilon + \varepsilon g_\infty, \tag{3.45}$$

with $\varepsilon^{-1}(\rho_+ - 1)$ bounded. Therefore the asymptotic values of z and w are

$$z(-\infty) = w(x = -\infty) = 0, \tag{3.46}$$

$$z(\infty) = z_\infty \equiv -\varepsilon^{-1}\langle(\xi_1 - s)\phi_\varepsilon, f_\infty\rangle = -\langle(\xi_1 - s)\psi_\varepsilon, f_\infty\rangle, \tag{3.47}$$

$$w(x = \infty) = w_\infty(\xi) = \varepsilon^{-1}(I - \Pi)f_\infty = (I - \Pi)g_\infty. \tag{3.48}$$

Since $\langle(\xi_1 - s)\phi_0, g_\infty\rangle = 0$, w_∞ is bounded; this justifies the scaling of w in (3.28).

Relations between z_∞ and w_∞ are found by applying P and $(I-P)$ to the equation $-\dfrac{1}{\varepsilon}Lf_\infty + \nu\Gamma(f_\infty, f_\infty) = 0$ to obtain

$$-\tau z_\infty + \gamma z_\infty^2 + \varepsilon\langle \mu_\varepsilon, z_\infty\psi_\varepsilon + w_\infty\rangle - \varepsilon\langle \psi_\varepsilon, \nu\Gamma(2\phi_\varepsilon z_\infty + \varepsilon w_\infty, w_\infty)\rangle = 0, \tag{3.49}$$

$$-\varepsilon^{-1}Mw_\infty - M_3 w_\infty - z_\infty(I-P)\mu_\varepsilon + \varepsilon^{-1}(I-P)\nu\Gamma(z_\infty\phi_\varepsilon + \varepsilon w_\infty, z_\infty\phi_\varepsilon + \varepsilon w_\infty)\} = 0. \tag{3.50}$$

Define τ_0 and τ' by

$$\tau_0 = z_\infty\gamma = \tau - \varepsilon\tau' < 0. \tag{3.51}$$

We write $z = z_0 + \varepsilon z_1$ and $w = w_0 + w_1$. The function z_0 is chosen to be the dominant part of z with its complete asymptotic values; w_0 is artificially picked to assume the asymptotic values of w. The equations for these functions are

$$\frac{\partial}{\partial x} z_0 = -\tau_0 z_0 + \gamma z_0^2, \tag{3.52}$$

$$w_0 = \tfrac{1}{2}\{\tanh(-\tfrac{1}{2}\tau_0 x) + 1\}w_\infty, \tag{3.53}$$

$$\frac{\partial}{\partial x} z_1 = -\tau z_1 + 2\gamma z_0 z_1 + a, \tag{3.54}$$

$$(\xi_1 - s)\frac{\partial}{\partial x} w_1 = -\frac{1}{\varepsilon}Mw_1 + \frac{1}{\varepsilon}b, \tag{3.55}$$

with

$$a = -\tau' z_0 + \varepsilon\delta z_1^2 + \langle \mu_\varepsilon, z\psi_\varepsilon + w\rangle - \langle \psi_\varepsilon, \nu\Gamma(2\phi_\varepsilon z + \varepsilon w, w)\rangle, \tag{3.56}$$

$$\begin{aligned} b = {} & -\varepsilon(\xi_1 - s)\frac{\partial}{\partial x} w_0 - Mw_0 - \varepsilon M_3 w + \varepsilon z(I-P)\mu_\varepsilon \\ & + (I-P)\nu\Gamma(z\phi_\varepsilon + \varepsilon w, z\phi_\varepsilon + \varepsilon w). \end{aligned} \tag{3.57}$$

The solution of (3.52) is

$$z_0(x) = \tfrac{1}{2}\{\tanh(-\tfrac{1}{2}\tau_0 x) + 1\}z_\infty, \tag{3.58}$$

which is unique up to a shift in x. Comparison of (3.56) and (3.57) with (3.49) and (3.50) shows that the asymptotic values of z_1 and w_1 are

$$z_1(-\infty) = z_1(\infty) = w_1(-\infty) = w_1(\infty) = 0, \tag{3.59}$$

since $a = b = 0$ at $x = \pm\infty$ with these values of z_1, w_1. The solution of (3.54) and (3.55) occupies the remainder of this paper.

We can see the necessity of the entropy condition (2.12) by considering a solution with $s(1-\rho) < 0$. Then we would have $s > c_0$, which would change the sign of a number of terms in the previous section. The result would be that $\tau > 0$ and the solution of (3.52) would not assume the required asymptotic values.

To check agreement with the Navier–Stokes profile we need only show that the shock widths in (2.21) and (3.58) are identical, i.e. $\tau_0 = \frac{3}{2}(\eta + \frac{1}{5}\lambda)^{-1}$. One can show that $F_1 = -(\tau_0 c_0)^{-1}\phi_1' u_x$; when substituted in (2.20) this shows that $\frac{4}{3}c_0(\eta + \frac{1}{5}\lambda) = -\langle \phi_0', (\tau_0 c_0)^{-1}(\xi_1 - c_0)\phi_1' \rangle = (\tau_0 c_0)^{-1}\langle \phi_0', \phi_0' \rangle$ which verifies that identity.

4. Basis Estimates

In this section we prove basic estimates which will be used in Sect. 5–7 to analyze Eqs. (3.54) and (3.55). First define the characteristic functions

$$\chi_N = \chi(|\xi| < N),$$
$$\chi_\delta = \chi(|\xi_1 - s| < \delta), \tag{4.1}$$

and the (generalized) resolvents

$$R_\lambda = (\lambda(\xi_1 - s) + M)^{-1}, \tag{4.2}$$

$$S_\lambda = (\lambda(\xi_1 - s) + \nu(\xi))^{-1}, \tag{4.3}$$

which act on $G_{\alpha,r}$ and

$$R_{N\lambda} = (\chi_N(\lambda(\xi_1 - s) + M)\chi_N)^{-1}, \tag{4.4}$$

which acts on $G_{\alpha,r}(|\xi| < N)$. The following five propositions are the main results of this section. In each of them we assume that

$$r \in R^1, \quad s \in R^1, \quad 0 \leqq \gamma \leqq 1, \quad 0 \leqq \alpha \leqq \tfrac{1}{4}, \quad 0 \leqq \theta \leqq 1. \tag{4.5}$$

Constant factors are omitted from the following estimates. They are uniformly bounded in any closed, bounded set of the parameters $(r, s, \gamma, \alpha, \theta)$ satisfying (4.5) and are inessential.

Proposition 4.1. Resolvent Estimates. *Let* $|\mathrm{Re}\,\lambda| < 2|\mathrm{Im}\,\lambda|$, *then*

$$\|(\xi_1 - s)^\theta R_\lambda h\|_{\alpha,r} \leqq |\lambda|^{-\theta}\|h\|_{\alpha,r-\gamma(1-\theta)}, \tag{4.6}$$

$$\|R_\lambda(\xi_1 - s)^\theta h\|_{\alpha,r} \leqq |\lambda|^{-\theta}\|h\|_{\alpha,r-\gamma(1-\theta)}, \tag{4.7}$$

If $|\mathrm{Re}\lambda| < 2|\mathrm{Im}\lambda|$ *or if* $|\mathrm{Re}\lambda| < \frac{1}{2}\nu_1 N^{\gamma-1}$, *then the same estimates are true for* $R_{N\lambda}$.

Proposition 4.2. Estimates on $\mathbf{S}_\lambda$. *If* $|\mathrm{Re}\,\lambda| < 2|\mathrm{Im}\,\lambda|$ *and* $0 \leqq \theta \leqq 1$, *then*

$$|\lambda^\theta(\xi_1 - s)^\theta S_\lambda| \leqq (1 + \xi)^{-\gamma(1-\theta)}, \tag{4.8}$$

$$|(\xi_1 - s)^\theta S_\lambda| \leqq (1 + \xi)^{\theta - \gamma}. \tag{4.9}$$

If $|\mathrm{Re}\,\lambda| < 2|\mathrm{Im}\,\lambda|$ *or if* $|\mathrm{Re}\,\lambda| < \dfrac{\nu_1}{2}N^{\gamma-1}$, *then the same estimates are true for* $\chi_N S_\lambda$.

Proposition 4.3. Estimates on H.

$$\|Hh\|_{\alpha,r+2-\gamma} \leqq \|h\|_{\alpha,r}, \tag{4.10}$$

$$\|H\chi_\delta((\xi_1 - s)^{-\theta}h)\|_{\alpha,r+2-\gamma} \leqq \delta^{1-\theta}\|h\|_{\alpha,r}. \tag{4.11}$$

If $|\mathrm{Re}\,\lambda| < 2|\mathrm{Im}\,\lambda|$,

$$\|HS_\lambda h\|_{\alpha,r+2-\gamma} \leqq (1+|\lambda|)^{-1/2}\|h\|_{\alpha,r}. \tag{4.12}$$

If $|\mathrm{Re}\,\lambda| < 2|\mathrm{Im}\,\lambda|$ or $|\mathrm{Re}\,\lambda| < \frac{v_1}{2}N^{\gamma-1}$,

$$\|H\chi_N S_\lambda h\|_{\alpha,r+2-\gamma} \leqq (1+\lambda)^{-1/2}\|h\|_{\alpha,r}. \tag{4.13}$$

Proposition 4.4. Sup Compactness of H. (i) *If* $\|f_n\|_{\alpha,r} \leqq 1$ *for all* n, *then* Hf_n *has a subsequence which converges in* $G_{\alpha,r+2-\gamma-\varepsilon_1}$, *for any* $\varepsilon_1 > 0$.

(ii) *The same is true for* $H(\xi_1 - s)^{-\theta}$.

Proposition 4.5. Bounds on the Integral Kernel.

The integral kernel $\bar{k}$ *from (3.40) satisfies*

$$\int_{|\eta_1 - s| \leqq \delta} |\eta_1 - s|^{-\theta}\bar{k}(\xi,\eta)(1+\eta)^{-r}e^{-\alpha\eta^2}d\boldsymbol{\eta} \leqq \delta^{1-\theta}(1+\xi)^{-r-2+\gamma}e^{-\alpha\xi^2}, \tag{4.14}$$

$$|\bar{k}(\xi,\eta)| \leqq v^{-1}(1+\xi+\eta)^{-1+\gamma}\exp\{-(1-\beta_1)(\tfrac{1}{8}v^2 + \tfrac{1}{2}\zeta_1^2)\}, \tag{4.15}$$

for any $0 < \beta_1 < 1$, *in which* $\mathbf{v} = \boldsymbol{\xi} - \boldsymbol{\eta}$ *and* ζ_1 *is the component of* $\frac{1}{2}(\boldsymbol{\xi} + \boldsymbol{\eta})$ *parallel to* $\boldsymbol{\xi} - \boldsymbol{\eta}$, $\zeta_1^2 = \frac{1}{4}(2\boldsymbol{\xi}\cdot\mathbf{v} + v^2)^2 v^{-2}$.

The properties are all still true if the modified operators M and H are replaced by the original operators L and K after the null space L is removed. Grad [9] proved the compactness of K as an operator on L^2, which can be used to show its compactness in sup norm; the refined bounds (4.14) and (4.15) enable us to prove it in $G_{\alpha,r}$ even including factors of $(\xi_1 - s)^{-\theta}$. This leads to the strong estimates (4.6) and (4.7). In Propositions 4.1, 4.2, and 4.3, the factors 2 and 1/2 could be replaced by numbers with magnitude > 1 and < 1 respectively. The proofs of these propositions will be presented in five subsections.

A. The Integral Kernel

Proof of (4.15). This was already proved for $k(\xi,\eta)$ in Proposition 5.1 of [2] (there is a change of sign in the definition of γ). The functions χ_i, ϕ_0 and ϕ_1 are all in $G_{1/4-3}$ according to (2.9) and Lemma 3.1. Then $L\phi_1 = v(\xi)\phi_1 + K\phi_1 \in G_{1/4-\beta_1,0}$ for any $\beta_1 > 0$ by Proposition 6.1 of [2]. This shows that each term in the expression (3.40) for $\bar{k}$ satisfies (4.15).

Next we state some auxiliary lemmas.

Lemma 4.6.

$$v^2 + 4\zeta_1^2 - 2\xi^2 + 2\eta^2 > 0, \tag{4.16}$$

$$\int_0^\infty e^{-x^2-2xy}dx \leqq \frac{1}{2y} \quad \text{for} \quad y > 0, \tag{4.17}$$

$$\int_{-1}^{1} (1-x^2)^{-1/2}dx \leqq c. \tag{4.18}$$

Lemma 4.7. (i) *Let* ξ', v', ξ_1, v_1 *all be* $\geqq 0$. *Denote* $g(x) = \exp\{-\kappa(2x\xi'v' + 2\xi_1 v_1 + v'^2 + v_1^2)^2/(v_1^2 + v'^2)\}$, *then*

$$\int_{-1/2}^{1/2} (1-x^2)^{-1/2} g(x)\,dx \leqq c\{(v_1^2 + v'^2)^{1/2}\kappa^{-1/2}/(\xi'v' + 1) + 1\}. \tag{4.19}$$

(ii) *If* $\max\{\xi_1 v_1, v'^2, v_1^2\} < \frac{1}{8}\xi'v'$, *then*

$$\int_{1 \geqq |x| \geqq 1/2} (1-x^2)^{-1/2} g(x)\,dx \leqq c\exp\{-\tfrac{1}{4}\kappa(\xi'v')^2/(v_1^2 + v'^2)\}$$
$$\leqq c(1 + \xi'v'(v_1^2 - v'^2)^{-1/2})^{-1}. \tag{4.20}$$

Proof. Omit $(1-x^2)^{-1/2}$ and integrate over all x to get (4.19). To prove (4.20) estimate $|2x\xi'v' + 2\xi_1 v_1 + v'^2 + v_1^2| > \frac{1}{2}|\xi'v'|$ for $|x| > \frac{1}{2}$.

Proof of (4.14). Denote $\boldsymbol{\xi} = (\xi_1, \boldsymbol{\xi}')$, $\eta = (\eta_1, \boldsymbol{\eta}')$, $\mathbf{v} = \boldsymbol{\xi} - \boldsymbol{\eta} = (v_1, \mathbf{v}')$ in which $\boldsymbol{\xi}' = (\xi_2, \xi_3)$, etc. Then

$$\boldsymbol{\xi}' \cdot \mathbf{v}' = x\xi'v' \quad \text{with} \quad x = \cos\measuredangle(\boldsymbol{\xi}', \mathbf{v}'), \tag{4.21}$$
$$d\mathbf{v} = v'(1-x^2)^{-1/2}\,dv'\,dx,$$
$$\eta^2 = \eta_1^2 + \xi'^2 + v'^2 + 2x\xi'v'. \tag{4.22}$$

If $\alpha < \frac{1}{4}$, it follows from (4.15) and (4.16) that

$$\begin{aligned}\tilde{k}(\boldsymbol{\xi}, \boldsymbol{\eta}) &\equiv |\eta_1 - s|^{-\theta}(1+\eta)^{-r}(1+\xi)^{1+r-\gamma} e^{\alpha(\xi^2 - \eta^2)} \bar{k}(\boldsymbol{\xi}, \boldsymbol{\eta}) \\ &\leqq |\eta_1 - s|^{-\theta} v^{-1} \exp\{-\kappa(v^2 + 4\zeta_1^2)\} \\ &\leqq |\eta_1 - s|^{-\theta}(v_1^2 + v'^2)^{-1/2} \exp\{-\kappa(v_1^2 + v'^2 \\ &\quad + (2x\xi'v' + 2\xi_1 v_1 + v'^2)(v_1^2 + v'^2)^{-1})\},\end{aligned} \tag{4.23}$$

for any $\kappa < \frac{1}{4} - \alpha$ (there is a constant depending on κ which has been omitted). Using the notation of Lemma 4.7, we write

$$\int_{|\eta_1 - s| \leqq \delta} \tilde{k}(\boldsymbol{\xi}, \boldsymbol{\eta})\,d\boldsymbol{\eta} \leqq \int_{|\eta_1 - s| \leqq \delta} d\eta_1 \int_0^\infty dv' \int_{-1}^{1} dx\, |\eta_1 - s|^{-\theta}(1-x^2)^{1/2}$$
$$\cdot v'(v_1^2 + v'^2)^{-1/2} \exp\{-\kappa(v_1^2 + v'^2)\} g(x). \tag{4.24}$$

Now use Lemma 4.7 to estimate this

(1) Let $\Omega_1 = \Omega_1(\boldsymbol{\xi}) = \{\boldsymbol{\eta} : |\eta_1 - s| < \delta$ and $\max\{v'^2, v_1^2\} > \frac{1}{8}\xi'v'\}$. Then

$$\begin{aligned}\int_{\Omega_1} \tilde{k}(\boldsymbol{\xi}, \boldsymbol{\eta})\,d\boldsymbol{\eta} &\leqq \int_{|\eta_1 - s| \leqq \delta} d\eta_1 \int_0^\infty dv' |\eta_1 - s|^{-\theta} \cdot \exp\{-\kappa(v_1^2 + v'^2)\} \\ &\leqq \int_{|\eta_1 - s| \leqq \delta} |\eta_1 - s|^{-\theta} e^{-\kappa v_1^2} d\eta_1 \cdot \int_0^\infty e^{-\frac{1}{2}\kappa(v'^2 + \xi'v')} dv' \\ &\leqq \delta^{1-\theta} e^{-\kappa\xi_1^2/2}(1+\xi')^{-1} \\ &\leqq \delta^{1-\theta}(1+\xi)^{-1},\end{aligned} \tag{4.25}$$

by dropping the terms $v'(v_1^2 + v'^2)^{-1/2} < 1$ and $(2x\xi'v' + v'^2 + v_1^2)^2(v_1^2 + v'^2)^{-1} > 0$

in the first step and using (4.17) and the identity $v_1 = \xi_1 - \eta_1$ in the third step.

(2) Let $\Omega_2 = \{\boldsymbol{\eta}: |\eta_1 - s| < \delta$ and $\xi_1 v_1 > \frac{1}{8}\xi' v'\}$. In this set $v_1^2 > \frac{1}{8}\xi' v' - \eta_1 v_1$. As above estimate

$$\begin{aligned}\int_{\Omega_2}^{\infty} \tilde{k}(\boldsymbol{\xi},\boldsymbol{\eta})d\eta &\leqq \int_{|\eta_1 - s| \leqq \delta} d\eta_1 \int_0^{\infty} dv' |\eta_1 - s|^{-\theta} \\ &\quad \cdot \exp\{-\tfrac{1}{2}\kappa(v_1^2 + v'^2 + \tfrac{1}{8}\xi' v' - \eta_1 v_1)\} \\ &\leqq \int_{|\eta_1 - s| \leqq \delta} e^{-\frac{1}{2}\kappa(v_1^2 - \eta_1 v_1)} |\eta_1 - s|^{-\theta} d\eta_1 \\ &\quad \cdot \int_0^{\infty} \exp\{-\tfrac{1}{2}\kappa(v'^2 + \tfrac{1}{8}\xi' v')\} dv' \\ &\leqq \delta^{1-\theta} e^{-\frac{3}{8}\kappa\xi_1^2}(1+\xi')^{-1} \\ &\leqq \delta^{1-\theta}(1+\xi)^{-1}. \end{aligned} \tag{4.26}$$

(3) Let $\Omega_3 = \{\boldsymbol{\eta}: |\eta_1 - s| < \delta$ and $\max\{\xi_1 v_1, v'^2, v_1^2\} < \frac{1}{8}\xi' v'$. Use (4.19) and (4.20) to derive

$$\begin{aligned}\int_{\Omega_3} \tilde{k}(\xi,\eta)d\eta &\leqq \int_{|\eta_1 - s| \leqq \delta} d\eta_1 \int_0^{\infty} dv' |\eta_1 - s|^{-\theta} v'(v_1^2 + v'^2)^{-1/2} \\ &\quad \cdot (1 + \xi' v'(v_1^2 + v'^2)^{-1/2})^{-1} e^{-\kappa(v_1^2 + v'^2)} \\ &\leqq \int_{|\eta_1 - s| \leqq \delta} |\eta_1 - s|^{-\theta} e^{-\kappa v_1^2} d\eta_1 \int_0^{\infty} (1+\xi')^{-1} e^{-\kappa v'^2} dv' \\ &\leqq \delta^{1-\theta} e^{-\frac{1}{2}\kappa\xi_1^2}(1+\xi')^{-1} \\ &\leqq \delta^{1-\theta}(1+\xi)^{-1}. \end{aligned} \tag{4.27}$$

Finally (4.14) follows from (4.25), (4.26), (4.27) and the definition (4.23) of $\tilde{k}$.

B. Estimates on S_λ

Lemma 4.8. *If $a \geqq 0$, $b > 0$, $1 \geqq \theta \geqq 0$, then*

$$(a+b)^{-1} a^{\theta} \leqq b^{\theta - 1}. \tag{4.28}$$

If $a \geqq 0$, $b \geqq (a/\lambda)^{\gamma}$, $\theta \geqq 0$, then

$$(a+b)^{-1} a^{\theta} \leqq \lambda^{\theta} b^{\theta/\gamma - 1}. \tag{4.29}$$

Lemma 4.9. *If $|\mathrm{Im}\,\lambda| > 2|\mathrm{Re}\,\lambda|$ or if $\xi < N$ and $|\mathrm{Re}\,\lambda| < \frac{1}{2}v_1 N^{\gamma-1}$, then*

$$|\lambda(\xi_1 - s) + v| > \tfrac{1}{8}(|\lambda(\xi_1 - s)| + v). \tag{4.30}$$

Proof of Proposition 4.2. It follows from (2.8), (4.28) and (4.30) that

$$\begin{aligned}|(\lambda(\xi_1 - s))^{\theta} S_\lambda| &\leqq 8|\lambda(\xi_1 - s)|^{\theta}(|\lambda(\xi_1 - s)| + v)^{-1} \\ &\leqq 8 v_2^{-1+\theta}(1+\xi)^{-\gamma(1-\theta)}, \end{aligned} \tag{4.31}$$

which is exactly (4.8) (with constants omitted). Since for ξ large $v(\xi) > v_1|\lambda(\xi_1 - s)/\lambda|^{\gamma}$, (4.9) follows from (4.29) in the same way. The estimates on $\chi_N S_\lambda$ are proved analogously.

C. Estimates on H: Proof of Proposition 4.3

As in Proposition 6.1 of [2], (4.10) follows from (4.15). From (4.14), (4.11) easily follows. We only need to prove (4.12) and (4.13) for large λ; otherwise they follow from (4.10) and (4.9) with $\theta=0$. Denote $\chi_1=\chi(|\xi_1-s|<|\lambda|^{-1/2})$ and $\chi_2=1-\chi_1$. It follows from (4.14) and (4.9) that

$$\|H\chi_1 S_\lambda h\|_{\alpha,r+2-\gamma}\leqq|\lambda|^{-1/2}\|h\|_{\alpha,r-\gamma}. \tag{4.32}$$

But if $|\xi_1-s|>\lambda^{-1/2}$, then $|\lambda(\xi_1-s)+v|>\frac{1}{8}(|\lambda(\xi_1-s)|+v)>\frac{1}{8}|\lambda|^{1/2}$ by (4.30), so that

$$\|H\chi_2 S_\lambda h\|_{\alpha,r+2-\gamma}\leqq|\lambda|^{-1/2}\|h\|_{\alpha,r}, \tag{4.33}$$

using (4.10). This proves (4.12); (4.13) is proved the same way.

D. Compactness of H

The compactness of H comes from continuity properties of its kernel. First we prove continuity for k, then for $\bar{k}$. The formula for k is given in (A.5)–(A.7), and we use that notation in the following. We also abbreviate "*locally uniformly continuous*" by LUC.

Lemma 4.10. $I_1(\xi,\boldsymbol{\eta})=\int\exp\{-\frac{1}{2}|\mathbf{w}+\zeta_2|^2\}q(\mathbf{v},\mathbf{w})d\mathbf{w}$ is LUC in ξ and $\boldsymbol{\eta}$.

Proof. The integrand is LUC in $\boldsymbol{\xi},\boldsymbol{\eta},\mathbf{w}$, but the domain of integration $\{w\perp(\boldsymbol{\xi}-\boldsymbol{\eta})\}$ is infinite and changes continuously as $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ change. So the integral $I_2(\xi,\boldsymbol{\eta})$ over $|w|<N$ is LUC for any N.

Now fix $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ and let $\varepsilon_1>0$. Pick N large enough that $\xi<N$, $\eta<N$, and $e^{-\frac{1}{4}N^2}<\varepsilon_1$. Then $\zeta_2<N, v<2N$ and

$$\begin{aligned} I_3(\xi,\boldsymbol{\eta}) &= \int_{w\geqq 2N}\exp\{-\tfrac{1}{2}|\mathbf{w}+\zeta_2|^2\}q(\mathbf{v},\mathbf{w})d\mathbf{w} \\ &\leqq N^\gamma e^{-(1/4)N^2}, \end{aligned} \tag{4.34}$$

using (A.9). Thus if $|\boldsymbol{\xi}-\boldsymbol{\xi}_1|<\delta$ and $|\boldsymbol{\eta}-\boldsymbol{\eta}_1|<\delta$, with δ small enough,

$$\begin{aligned} |I_1(\xi_1,\eta_1)-I_1(\xi,\eta)| &< |I_3(\xi_1,\eta_1)-I_3(\xi,\boldsymbol{\eta})| \\ &\quad+|I_2(\xi_1,\boldsymbol{\eta}_1)-I_2(\xi,\boldsymbol{\eta})| \\ &< 2\varepsilon_1. \end{aligned} \tag{4.35}$$

Lemma 4.11. *For any* $\varepsilon_1>0$, $\alpha<\frac{1}{4}$, *and any* r

$$\int(1+\eta)^{-r+2-\gamma-\varepsilon_1}e^{-\alpha\eta^2}|(1+\xi)^r e^{\alpha\xi^2}k(\xi,\boldsymbol{\eta})-(1+\hat{\xi})^r e^{\alpha\hat{\xi}^2}k(\hat{\xi},\boldsymbol{\eta})|\,d\boldsymbol{\eta}\to 0, \tag{4.36}$$

as $\hat{\xi}\to\xi$ *locally uniformly in* ξ.

Proof. Denote

$$h = (1+\xi)^r (1+\eta)^{-r+2-\gamma-\varepsilon_1} e^{\alpha(\xi^2-\eta^2)} k(\xi,\eta). \tag{4.37}$$

According to (4.15) for k instead of $\bar{k}$,

$$\begin{aligned} &\int |h| d\eta \leqq (1+\xi)^{-\varepsilon_1}, \\ &\int_{\eta>N} |h| d\eta \leqq (1+N)^{-\varepsilon_1}, \end{aligned} \tag{4.38}$$

which shows that large ξ and η can be ignored. Let $\varepsilon_2 > 0$ and pick N large enough that $N^{-\varepsilon_1} < \varepsilon_2$. According to (4.37), (A.7) and Lemma 4.10, $h = g_1 + v^{-2} g_2$ in which g_1 and g_2 are LUC and uniformly bounded. The integral in (4.36) is

$$\begin{aligned} \int |h(\hat{\xi},\eta) - h(\xi,\eta)| d\eta &\leqq 2N^{-\varepsilon_1}, \quad \text{if} \quad \xi > N, \\ &\leqq 2N^{-\varepsilon_1} + \int_{\eta<N} |h(\hat{\xi},\eta) - h(\xi,\eta)| d\eta, \quad \text{if} \quad \xi < N. \end{aligned} \tag{4.39}$$

Estimate

$$\begin{aligned} \int_{\eta \leqq N} |h(\hat{\xi},\eta) - h(\xi,\eta)| d\eta \leqq \int_{\eta \leqq N} \{ &|g_1(\hat{\xi},\eta) - g_1(\xi,\eta)| + |\xi-\eta|^{-2} |g_2(\hat{\xi},\eta) - g(\xi,\eta)| \\ &+ |g_2(\xi,\eta)| \cdot ||\xi-\eta|^{-2} - |\hat{\xi}-\eta|^{-2}| \} d\eta. \end{aligned} \tag{4.40}$$

By the fact that g_1 and g_2 are LUC and the integrability of $|\xi-\eta|^{-2}$ in $\eta < N$, the first two integrals go to 0 as $\xi_1 \to \xi$. Furthermore $\int_{\eta<N} d\eta ||\xi-\eta|^{-2} - |\hat{\xi}_1 - \hat{\eta}|^{-2}| \leqq cN^{-3}$, and g_2 is uniformly bounded. Therefore the integral in (4.39) can be made arbitrarily small by first taking N large, then $|\xi - \hat{\xi}_1|$ small, which proves (4.36).

We say that $h(\xi)$ is LUC in $G_{\alpha,r}$ if $h \in G_{\alpha,r}$ and

$$(1+\xi)^r e^{\alpha\xi^2} h(\xi) - (1+\hat{\xi}) e^{\widehat{\alpha\xi^2}} h(\hat{\xi}) \to 0, \tag{4.41}$$

as $\hat{\xi} \to \xi$ locally uniformly in ξ.

Lemma 4.12. *(i) If $\psi(\xi)$ is LUC in $G_{\alpha,r}$, then $L\psi(\xi)$ is LUC in $G_{\alpha,r+\gamma}$; (ii) If $\psi(\xi)$ is LUC in $G_{\alpha,r}$, and $\kappa(\xi) \in G_{\alpha,r+\gamma}$ with $L\kappa = \psi$, then κ is LUC in $G_{\alpha,r+\gamma}$.*

Proof. Since $v(\xi)$ in LUC, $v(\xi)\psi(\xi)$ is LUC in $G_{\alpha,r+\gamma}$. By (4.36)

$$\begin{aligned} (1+\hat{\xi})^r e^{\widehat{\alpha\xi^2}} K\psi(\hat{\xi}) - (1+\xi)^r e^{\alpha\xi^2} K\psi(\xi) &\leqq \|\psi\|_{\alpha,r} \int (1+\eta)^{-r} e^{-\alpha\eta^2} | \\ \cdot (1+\hat{\xi})^{r+2-\gamma-\varepsilon_1} e^{\alpha\hat{\xi}^2} k(\hat{\xi},\eta) - &(1+\xi)^{r+2-\gamma-\varepsilon_1} e^{\alpha\xi^2} k(\xi,\eta) | d\eta \to 0, \end{aligned} \tag{4.42}$$

as $\hat{\xi} \to \xi$ locally uniformly. Since $2 > \gamma + \varepsilon_1$, this shows that $Lu = vu - Ku$ is LUC in $G_{\alpha,r+\gamma}$.

As in (4.42) we have $K\kappa$ LUC in $G_{\alpha,r+\gamma}$. Thus $\kappa(\xi) = v(\xi)^{-1} \cdot (K\kappa(\xi) + \psi(\xi))$ is LUC in $G_{\alpha,r+\gamma}$.

Proposition 4.13. *The kernel $\bar{k}(\xi,\eta)$ satisfies (4.36).*

Proof. The functions χ_i $(i = 0, \ldots, 4)$ and ϕ_0 are LUC in $G_{1/4,-2}$. By Lemma 4.12,

ϕ_1 is LUC in $G_{1/4,\gamma-3}$. It follows from Lemma 4.12 that every term in the expression (3.40) for $\bar{k}+k$ is LUC in $G_{\alpha,r}$ for any $\alpha<\frac{1}{4}$ and any r. An easy estimate shows that $\bar{k}+k$ satisfies (4.36) (for k replaced by $\bar{k}+k$) and hence so does $\bar{k}$.

Finally a use of the inequality (4.42) with k replaced by $\bar{k}$ shows

Lemma 4.14. *Let* $\|f\|_{\alpha,r}<c$, *then* Hf *is* LUC *in* $G_{\alpha,r+2-\gamma-\varepsilon_1}$, *with a modulus of continuity which depends on* c.

To show compactness we shall employ the following version of Arzela–Ascoli theorem.

Lemma 4.15. Arzela–Ascoli. *Let* $h_n(x)$, $x\in R^m$ *be a sequence of functions with*

(i) $|h_n(x)|\leqq g(|x|)$ *with* $g(|x|)\to 0$ *as* $|x|\to\infty$ and g uniformly bounded; (ii) h_n *uniformly* (*in* n) *equi-continuous, locally uniformly* (*in* x), i.e. *for each* $\varepsilon>0$, $N>0$, $\exists\ \delta>0$ *such that if* $|x|<N$, $|x-y|<\delta$, *then* $|h_n(x)-h_n(y)|<\varepsilon$.

Then f_n *has a subsequence which converges uniformly* (*in* x).

Proof of (i) *in Proposition 4.4.* Let $\|f_n\|_{\alpha,r}\leqq c$. By (4.10), $h_n\doteq(1+\xi)^{r+2-\gamma-\varepsilon_1}\cdot Hf_n(\xi)<c(1+\xi)^{-(1/2)\varepsilon_1}$ for any $\varepsilon_1>0$. Thus h_n satisfies the uniform bound required by (i) of Lemma 4.15; by Lemma 4.14 it also satisfies the continuity requirements of (ii). Therefore Hf_n has a convergent subsequence in $G_{\alpha,r+2-\gamma-\varepsilon_1}$.

Proof of (*ii*) *in Proposition 4.4.* Denote $\chi_\delta=\chi(|\xi_1-s|<\delta)$ and $\bar{\chi}_\delta=1-\chi_\delta$. Let $\varepsilon>0$ and pick δ so small that

$$\|H\chi_\delta(\xi_1-s)^{-\theta}f_n\|_{\alpha,r+2-\gamma}\leqq\delta^{1-\theta}\leqq\varepsilon, \tag{4.43}$$

using (4.11). By part (i), $H\bar{\chi}_\delta(u+\xi_1)^{-\theta}f_n$ has a convergent subsequence with indices n_j so that if $j>N$, $i>N$, then $\|H\chi_\delta(\xi_1-s)^{-\theta}\cdot(f_{n_j}-f_{n_i})\|_{\alpha,r+2-\gamma-\varepsilon_1}\leqq\varepsilon$ and so the same is true of $H(\xi_1-s)^{-\theta}(f_{n_j}-f_{n_i})$. By a diagonalization procedure there is a subsequence of $H(\xi_1-s)^{-\theta}f_n$ which converges in $G_{\alpha,r+2-\gamma-\varepsilon_1}$.

E. Resolvent Estimates

Proof of (*4.7*) *in Proposition 4.1.* Suppose to the contrary that there are λ_n satisfying $|\mathrm{Re}\,\lambda_n|<2|\mathrm{Im}\,\lambda_n|$ and s_n, f_n, g_n such that s_n are uniformly bounded and

$$R_{\lambda_n}(\xi_1-s_n)^\theta f_n=g_n,$$
$$\|f_n\|_{\alpha,r-\gamma(1-\theta)}=1,$$
$$\|g_n\|_{\alpha,r}=\lambda_n^{-\theta}a_n,\qquad a_n\to\infty. \tag{4.44}$$

Denote $\kappa_n=a_n^{-1}f_n$ and $\psi_n=\lambda_n^\theta a_n^{-1}g_n$ so that

$$\|\psi_n\|_{\alpha,r}=1,$$
$$\lambda_n^\theta(\xi_1-s_n)^\theta\kappa_n=(\lambda_n(\xi_1-s_n)+v+H)\psi_n. \tag{4.45}$$

Multiply by S_{λ_n}, which is bounded according to (4.9), to get

$$\lambda_n^\theta(\xi_1-s_n)^\theta S_{\lambda_n}\kappa_n=\psi_n+S_{\lambda_n}H\psi_n. \tag{4.46}$$

From (4.8) and (4.44), it follows that

$$\|\lambda_n^{\theta}(\xi_1 - s_n)^{\theta} S_{\lambda_n} \kappa_n\|_{\alpha,r} \leqq \|\kappa_n\|_{\alpha,r-\gamma(1-\theta)} \to 0. \tag{4.47}$$

Also, $\|\psi_n\|_{\alpha,r} = 1$ so that by Proposition 4.4 there is a subsequence of ψ_n (which we again call ψ_n) with $H\psi_n \to \phi$ in $G_{\alpha,r+2-\gamma-\varepsilon}$ and hence also in $G_{\alpha,r}$. Since S_{λ_n} is bounded uniformly, $\psi_n + S_{\lambda_n}\phi \to 0$ in $G_{\alpha,r}$.

Now by taking a subsequence we may assume that $s_n \to s_\infty < \infty$. Also either $\lambda_n \to \lambda_\infty$ with $|\lambda_\infty| < \infty$ or $|\lambda_n| \to \infty$ after possibly taking a subsequence.

1. Suppose $\lambda_n \to \lambda_\infty \neq 0$. Then $|\mathrm{Re}\,\lambda_\infty| < 2|\mathrm{Im}\,\lambda_\infty|$ and $S_{\lambda_n}\phi \to S_{\lambda_\infty}\phi = (\lambda_\infty \cdot (\xi_1 - s) + v)^{-1}\phi = -\Psi$ in $G_{\alpha,r+\gamma}$ and so $\psi_n \to \Psi$ in $G_{\alpha,r}$. Since $H\psi_n \to \phi$, we must have $\phi = H\Psi$, i.e.

$$\lambda_\infty(\xi_1 - s_\infty)\Psi + v\Psi + H\Psi = 0. \tag{4.48}$$

Moreover $\|\Psi\|_{\alpha,r} = \lim_n \|\Psi_n\|_{\alpha,r} = 1$. But this is impossible, since $M = v + H$ is self-adjoint and positive $\langle \Psi^*, (v+H)\Psi\rangle > 0$, and λ_∞ is complex, s_∞ is real, and $\langle \Psi^*, \lambda_\infty(\xi_1 - s_\infty)\Psi\rangle$ is complex.

2. Suppose $|\lambda_n| \to \infty$. We show that $\psi_n \to 0$. Let ε_1 be small and write $\psi_n = \psi_n^1 + \psi_n^2$ with $\psi_n^1 = \psi_n|_{|\xi_1 - s_n| < \varepsilon_1}$. Then since $S_{\lambda_n} \to 0$ on $|\xi_1 - s_n| > \varepsilon_1$, $\|\psi_n^2\|_{\alpha,r} \to 0$. For n large enough $\|H\psi_n^2\|_{\alpha,r} < \varepsilon_1$. Also

$$\|H\psi_n^1\|_{\alpha,r} < \varepsilon_1, \tag{4.49}$$

by (4.45) and (4.11) with $\theta = 0$. It follows that $H\psi_n \to 0$, $\phi = 0$ and hence $\|\psi_n\|_{\alpha,r} \to 0$, which contradicts the fact that $\|\psi_n\|_{\alpha,r} = 1$.

3. Suppose that $\lambda_n \to 0$. We show that

$$\psi_n \to -v^{-1}\phi = \Psi, \tag{4.50}$$

which implies that $M\Psi = H\Psi + v\Psi = 0$. This is a contradiction since M is positive.

To show this we split $\phi = \phi^1 + \phi^2$, $\psi_n = \psi_n^1 + \psi_n^2$ with

$$\phi^1 = \phi|_{|\xi| \geqq A}, \qquad \psi_n^1 = \psi_n|_{|\xi| \geqq A}. \tag{4.51}$$

Then clearly $\psi_n^2 \to -\frac{1}{v}\phi^2$ in $G_{\alpha,r}$. But $\|S_{\lambda_n}\phi^1\|_{\alpha,r} \leqq A^{-2+\varepsilon_1}$ since $\phi^1 \in G_{\alpha,r+2-\gamma-\varepsilon_1}$ and $|S_\lambda| < (1+\xi)^{-\gamma}$. Similarly $\|v^{-1}\phi^1\|_r \leqq A^{-2+\varepsilon_1}$. By choosing n and A large enough we can make

$$\|\psi_n - v^{-1}\phi\|_r < \|\psi_n^2 - v^{-1}\phi^2\|_r + \|\psi_n^1 - S_{\lambda_n}\phi^1\|_r + \|S_{\lambda_n}\phi^1 - v^{-1}\phi^1\|_r, \tag{4.52}$$

as small as we please which shows (4.50), and finishes the proof of (4.7).

The rest of Proposition 4.1 is proved in a similar way.

5. Solution of the Linearized Lyapunov–Schmidt Equation

We shall solve the linearized Lyapunov–Schmidt equation

$$(\xi_1 - s)\frac{\partial}{\partial x} w = -Mw + h, \tag{5.1}$$

as an initial value problem integrating forward in x over that part of h corresponding to negative spectrum and backward in x over that part corresponding to positive spectrum. Define contours

$$\begin{aligned} \Gamma_+ &= \{\lambda = z \pm 2iz, z \geqq 0\}, \\ \Gamma_- &= \{\lambda = -z \pm 2iz, z \geqq 0\}, \end{aligned} \tag{5.2}$$

and operators

$$\begin{aligned} U_+(x) &= (2\pi i)^{-1} \int_{\Gamma_+} e^{\lambda x} R_\lambda d\lambda, \qquad \text{for } x < 0, \\ U_-(x) &= (2\pi i)^{-1} \int_{\Gamma_-} e^{\lambda x} R_\lambda d\lambda, \qquad \text{for } x > 0, \end{aligned} \tag{5.3}$$

$$W[h](x) = \int_{-\infty}^{x} U_-(x-z)h(z)dz + \int_{x}^{\infty} U_+(x-z)h(z)dz. \tag{5.4}$$

Theorem 5.1. *Let h be a continuous function of x with values in $G_{\alpha,r-\gamma}$ with $0 \leqq \alpha < \frac{1}{4}$. Then $w(x) = W[h](x)$ solves (5.1) and is continuous as a function of x with values in $G_{\alpha,r}$ satisfying*

$$\sup_x \|w(x)\|_{\alpha,r} \leqq \sup_x \|h(x)\|_{\alpha,r-\gamma}. \tag{5.5}$$

This is proved with the aid of

Proposition 5.2.

$$\lim_{x \uparrow 0} (\xi_1 - s)U_+(x) + \lim_{x \downarrow 0} (\xi_1 - s)U_-(x) = 1. \tag{5.6}$$

Proof of Proposition 5.2. We use the resolvent identity

$$R_\lambda = S_\lambda - R_\lambda H S_\lambda. \tag{5.7}$$

First estimate

$$\begin{aligned} \|(\xi_1 - s)R_\lambda H S_\lambda h\|_{\alpha,r} &\leqq \|(\xi_1 - s)^{1-\bar{\varepsilon}} R_\lambda H S_\lambda h\|_{\alpha,r+\bar{\varepsilon}} \\ &\leqq \lambda^{-(1-\bar{\varepsilon})} \|H S_\lambda h\|_{\alpha,r+\bar{\varepsilon}-\bar{\varepsilon}\gamma} \\ &\leqq (1+|\lambda|)^{-1/2} \lambda^{-(1-\bar{\varepsilon})} \|h\|_{\alpha,r-2+\gamma+\bar{\varepsilon}-\bar{\varepsilon}\gamma} \\ &\leqq (1+|\lambda|)^{-1/2} \lambda^{-(1-\bar{\varepsilon})} \|h\|_{\alpha,r-\gamma}, \end{aligned} \tag{5.8}$$

for $0 \leqq \gamma \leqq 1$, $\bar{\varepsilon}$ small. So this quantity is absolutely integrable along Γ_+ and

$$\int_{\Gamma_+} (\xi_1 - s)R_\lambda H S_\lambda h d\lambda = \int_{-i\infty}^{i\infty} (\xi_1 - s)R_\lambda H S_\lambda h d\lambda, \tag{5.9}$$

by a shift of contour. Next evaluate

$$(2\pi i)^{-1} \int_{\Gamma_+} (\xi_1 - s)e^{\lambda x} S_\lambda d\lambda = (2\pi i)^{-1} \int_{\Gamma_+} e^{\lambda x}(\lambda + (\xi_1 - s)v)^{-1} d\lambda. \tag{5.10}$$

For $x < 0$, the exponential factor assures absolute convergence and the contour can be closed (with arbitrary accuracy). There is a singularity at $\lambda = -(\xi_1 - s)^{-1}v$ if $\xi_1 - s < 0$, and no singularity if $\xi_1 - s > 0$. So the integral is $e^{(\xi_1 - s)^{-1}vx}$ or 0 in these two cases and

$$\lim_{x\uparrow 0}(2\pi i)^{-1}\int_{\Gamma_+}(\xi_1 - s)e^{\lambda x}S_\lambda d\lambda = \begin{cases} 1, & (\xi_1 - s) < 0, \\ 0, & (\xi_1 - s) > 0. \end{cases} \tag{5.11}$$

So

$$\lim_{x\uparrow 0}(\xi_1 - s)U_+(x) = \chi(\xi_1 - s < 0) + (2\pi i)^{-1}\int_{-i\infty}^{i\infty}(\xi_1 - s)R_\lambda HS_\lambda d\lambda, \tag{5.12}$$

and similarly

$$\lim_{x\downarrow 0}(\xi_1 - s)u_-(x) = \chi(\xi_1 - s > 0) - (2\pi i)^{-1}\int_{-i\infty}^{i\infty}(\xi_1 - s)R_\lambda HS_\lambda d\lambda, \tag{5.13}$$

with the minus sign coming from the orientation of Γ_- in the direction of decreasing imaginary part. The result (5.6) comes from combining these.

Proof of Theorem 5.1. First we prove (5.5) to show that W is well defined. Using (5.7) we rewrite

$$\int_x^\infty U_+(x-z)h(z)dz = \int_x^\infty dz\int_{\Gamma_+}e^{\lambda(x-z)}(S_\lambda - R_\lambda HS_\lambda)h(z)d\lambda(2\pi i)^{-1}. \tag{5.14}$$

(i) By differentiating we see that

$$\frac{\partial}{\partial\lambda}R_\lambda HS_\lambda = -R_\lambda(\xi_1 - s)R_\lambda HS_\lambda - R_\lambda HS_\lambda(\xi_1 - s)S_\lambda. \tag{5.15}$$

Use Proposition 4.1, 4.2, and 4.3 to estimate

$$\begin{aligned}\|R_\lambda HS_\lambda(\xi_1 - s)S_\lambda\|_{\alpha,r} &\leqq \lambda^{-1+\beta}(1+|\lambda|)^{-1/2}\|h\|_{\alpha,r-2+\beta(1-\gamma)} \\ &\leqq \lambda^{-1/2}\|h\|_{\alpha,r-\gamma},\end{aligned} \tag{5.16}$$

$$\begin{aligned}\|R_\lambda HS_\lambda(\xi_1 - s)S_\lambda h\|_{\alpha,r} &\leqq \lambda^{-1+\beta}(1+|\lambda|)^{-1/2}\|h\|_{\alpha,r-2+\beta(1-\gamma)} \\ &\leqq \lambda^{-1/2}\|h\|_{\alpha,r-\gamma},\end{aligned} \tag{5.17}$$

after choosing $\beta = 1/2$. Now use integration by parts to obtain

$$\begin{aligned}&\left\|\int_{x+1}^\infty dz\int_{\Gamma_+}d\lambda e^{-\lambda(x-z)}R_\lambda HS_\lambda h(z)\right\|_{\alpha,r} \\ &= \left\|\int_{x+1}^\infty dz(x-z)^{-1}\int_{\Gamma_+}d\lambda e^{\lambda(x-z)}\{R_\lambda(\xi_1 - s)R_\lambda HS_\lambda\right. \\ &\qquad\left. + R_\lambda HS_\lambda(\xi_1 - s)S_\lambda\}h(z)\right\|_{\alpha,r} \\ &\leqq c\sup_z\|h(z)\|_{\alpha,r-\gamma}\int_{x+1}^\infty dz(x-z)^{-1}\int_0^\infty e^{\lambda(x-z)}\lambda^{-1/2}d\lambda \\ &\leqq c\sup_z\|h(z)\|_{\alpha,r}.\end{aligned} \tag{5.18}$$

(ii) Estimate

$$\|R_\lambda H S_\lambda h\|_{\alpha,r} \leqq \|HS_\lambda h\|_{\alpha,r-\gamma}$$
$$\leqq (1+\lambda)^{-1/2} \|h\|_{\alpha,r-2}. \tag{5.19}$$

So

$$\left\| \int_x^{x+1} dx \int_{\Gamma_+} d\lambda e^{\lambda(x-z)} R_\lambda H S_\lambda h(z) \right\|_{\alpha,r}$$
$$\leqq \int_x^{x+1} dx \int_0^\infty d\lambda e^{\lambda(x-z)}(1+\lambda)^{-1/2} \sup_z \|h(z)\|_{\alpha,r-2}$$
$$\leqq c \sup_z \|h(z)\|_{\alpha,r-2}. \tag{5.20}$$

(iii) Finally by a contour integration

$$\int_{\Gamma_+} e^{\lambda(x-z)} S_\lambda d\lambda = \begin{cases} 2\pi i(\xi_1 - s)e^{-(\xi_1-s)^{-1}\nu(x-z)}, & \xi_1 - s < 0, \\ 0, \quad \xi_1 - s > 0, \end{cases} \tag{5.21}$$

and

$$\int_x^\infty (\xi_1 - s)e^{-(\xi_1-s)^{-1}\nu(x-z)} dz = \nu^{-1}, \qquad \text{if } (\xi_1 - s) < 0. \tag{5.22}$$

So

$$\left\| \int_x^\infty dx \int_{\Gamma_+} d\lambda e^{\lambda(x-z)} S_\lambda h(z) \right\|_{\alpha,r} \leqq c \sup \|h(z)\|_{\alpha,r-\gamma}. \tag{5.23}$$

Combining (5.18), (5.20), and (5.23) using (5.14) yields

$$\left\| \int_x^\infty dz U_+(x-z)h(z) \right\|_{\alpha,r} \leqq c \sup_z \|h(z)\|_{\alpha,r-\gamma}. \tag{5.24}$$

A similar inequality can be found for U_- to deduce (5.5).

(iv) Next we show that w solves (5.1). We can differentiate to get

$$(\xi_1 - s)\frac{\partial}{\partial x} w(x) = \lim_{z \uparrow x} (\xi_1 - s)U_-(x-z)h(z) + \lim_{z \downarrow x} (\xi_1 - s)U_+(x-z)h(z)$$
$$+ \int_{-\infty}^x dz \int_{\Gamma_-} d\lambda \lambda e^{\lambda(x-z)}(\xi_1 - s)R_\lambda h(z)$$
$$+ \int_x^\infty dx \int_{\Gamma_+} d\lambda \lambda e^{\lambda(x-z)}(\xi_1 - s)R_\lambda h(z). \tag{5.25}$$

By (5.6) and the continuity of h, the sum of the first two terms on the right is $h(x)$. A resolvent identity tells us that $\lambda(\xi_1 - s)R_\lambda = -MR_\lambda + I$. By deformation of contours we easily see that

$$\int_{\Gamma_-} e^{\lambda(x-z)} d\lambda = 0, \qquad x - z > 0,$$
$$\int_{\Gamma_+} e^{\lambda(x-z)} d\lambda = , \qquad x - z < 0, \tag{5.26}$$

and so the sum of the last two terms on the right side of (5.25) is:

$$-\int_{-\infty}^{x} dz \int_{\Gamma_-} d\lambda e^{\lambda(x-z)} M R_\lambda h(z) - \int_{x}^{\infty} dz \int_{\Gamma_+} d\lambda e^{\lambda(x-z)} M R_\lambda h(z) = -Mw. \quad (5.27)$$

Therefore

$$(\xi_1 - s)\frac{\partial}{\partial x} w(x) = h(x) - Mw. \quad (5.28)$$

6. Decay of the Linearized Solution

Define the decay function $A(x)$ as in (2.25).

Theorem 6.1. *Suppose that $h(x,\xi)$ is continuous in x as an element of $G_{r,\alpha}$ and that*

$$\sup_x \|h(x)\|_{\alpha,r-\gamma} \leqq c_0, \quad (6.1)$$

$$\|h(x)\|_r \leqq c_1 A(\varepsilon x). \quad (6.2)$$

Then w defined as in Theorem 5.1 is continuous in x as an element of $G_{\alpha,r}$ and satisfies

$$\sup_x \|w(x)\|_{\alpha,r} \leqq c c_0, \quad (6.3)$$

$$\|w(x)\|_r \leqq c(c_0 + c_1) A(\varepsilon x), \quad (6.4)$$

if $\mu < 2^{-2/(1-\gamma)}\alpha$.

The proof of this theorem depends on a proper choice of N, the cutoff used in Sect. 5, as a function of x. Choose

$$N(x) = (\mu/\alpha)^{1/2}|x|^{\beta/2}, \qquad \beta = 2(3-\gamma)^{-1}$$
$$\mu = \alpha^{(\gamma-1)/(\gamma-3)}(4/v_1)^\beta, \quad (6.5)$$

so that

$$\alpha N^2 = \mu|x|^\beta$$
$$\tfrac{1}{2} v_1 N^{\gamma-1} = 2\mu|x|^{\beta-1}. \quad (6.6)$$

We use two elementary lemmas

Lemma 6.2. *If N is large enough and if $\xi < N$,*

$$|(\xi_1 - s)^{-1} v(\xi)| > v_1 N^{\gamma-1}. \quad (6.7)$$

If also $x > 1$, $\xi_1 - s > 0$,

$$(\xi_1 - s)^{-1}\exp\{-(\xi_1 - s)^{-1} v(\xi) x\} < N^{-1} e^{-v_1 N^{\gamma-1} x}. \quad (6.8)$$

Lemma 6.3. If $0 \leqq \beta \leqq 1$,

$$|x|^{\beta-1}|x-z| + |z|^\beta \geqq |x|^\beta, \quad (6.9)$$

$$|x|^{\beta-1}|x-z| + \varepsilon|z| \geqq \min(|x|^\beta, \varepsilon|x|), \quad (6.10)$$

$$\int_x^\infty N^{\gamma-1}e^{-\nu_1 N^{\gamma-1}(z-x)}\max(e^{-\mu|z|^\beta},e^{-\varepsilon|z|})dz \leqq \max(e^{-\mu|x|^\beta},e^{-\varepsilon|x|}). \tag{6.11}$$

Proof of Theorem 6.1. The bound (6.3) was proved in Theorem 5.1. To prove (6.4), we split the velocity space into two parts: $\{|\xi|<N\}$ and $\{|\xi|>N\}$ and define

$$\begin{aligned}
\chi_N &= \chi(|\xi|<N),\\
\bar\chi_N &= \chi(|\xi|>N),\\
R_{N\lambda} &= (\chi_N(\lambda(\xi_1-s)+\nu+H)\chi_N)^{-1},\\
S_{N\lambda} &= \chi_N(\lambda(\xi_1-s)+\nu)^{-1},\\
\bar S_{N\lambda} &= \bar\chi_N(\lambda(\xi_1-s)+\nu)^{-1}.
\end{aligned} \tag{6.12}$$

We define $R_{N\lambda}$, $S_{N\lambda}$ and $\bar S_{N\lambda}$ on $G_{\alpha,r}(\xi\in R^3)$ in the natural way, i.e. $R_{N\lambda}f=g$ means that $\chi_N(\lambda(\xi_1-s)+\nu+H)g=f$ and $\operatorname{supp} g\subset\{|\xi|<N\}$. Then

$$(\lambda(\xi_1-s)+\nu+\chi_N H\chi_N)^{-1}=R_{N\lambda}+\bar S_{N\lambda}, \tag{6.13}$$

$$\begin{aligned}
R_\lambda &= (R_{N\lambda}+\bar S_{N\lambda})(1-(\bar\chi_N H+\chi_N H\bar\chi_N)R_\lambda)\\
&= R_{N\lambda}+\bar S_{N\lambda}-T_\lambda,
\end{aligned} \tag{6.14}$$

$$T_\lambda = R_{N\lambda}\chi_N H\bar\chi_N R_\lambda+\bar S_{N\lambda}\bar\chi_N H R_\lambda. \tag{6.15}$$

Integrate each of the three terms in (6.14):

(i) By contour integration

$$(2\pi i)^{-1}\int_x^\infty\int_{\Gamma_+} e^{\lambda(x-z)}\bar S_{N\lambda}\,d\lambda\,dz=\begin{cases}\nu^{-1} & \text{if } \xi_1-s<0,\quad |\xi|>N,\\ 0, & \text{otherwise,}\end{cases} \tag{6.16}$$

$$\begin{aligned}
\left\|(2\pi i)^{-1}\int_x^\infty\int_{\Gamma_+} e^{\lambda(x-z)}\bar S_{N\lambda}h(z)\,d\lambda\,dz\right\|_r &\leqq c\|\bar\chi_N h\|_{r-\gamma}\\
&\leqq cc_0e^{-\alpha N^2}\\
&\leqq cc_0e^{-\mu|x|^\beta}.
\end{aligned} \tag{6.17}$$

A similar estimate is proved for the integral over Γ_-.

$$\begin{aligned}
\frac{\partial}{\partial\lambda}T_\lambda = &-R_{N\lambda}(\xi_1-s)R_{N\lambda}\chi_N H\bar\chi_N R_\lambda-R_{N\lambda}\chi_N H\bar\chi_N R_\lambda(\xi_1-s)R_\lambda\\
&-\bar S_{N\lambda}(\xi_1-s)\bar S_{N\lambda}\bar\chi_N HR_\lambda-\bar S_{N\lambda}\bar\chi_N HR_\lambda(\xi_1-s)R_\lambda.
\end{aligned} \tag{6.18}$$

Now estimate using first (4.6) for $R_{N\lambda}$ and (4.8) for $\bar S_{N\lambda}$ with $\theta=0$, then the resolvent identity $R_\lambda=S_\lambda(I-HR_\lambda)$, and finally (4.12), that

$$\begin{aligned}
\|T_\lambda h\|_r &\leqq \|H\bar\chi_N R_\lambda h\|_{r-\gamma}+\|\bar\chi_N HR_\lambda h\|_{r-\gamma}\\
&\leqq \|HS_\lambda\bar\chi_N(I-HR_\lambda)h\|_{r-\gamma}+\|\bar\chi_N HS_\lambda(I-HR_\lambda)h\|_{r-\gamma}\\
&\leqq (+\lambda)^{-1/2}e^{-\alpha N^2}\|h\|_{\alpha,r-2}.
\end{aligned} \tag{6.19}$$

Similarly we find that

$$\left\|\frac{\partial}{\partial\lambda}T_\lambda h\right\|_r \leqq \lambda^{-1/2}e^{-\alpha N^2}\|h\|_{\alpha,r-1}. \tag{6.20}$$

These are integrated as in the proof of Theorem 5.1. Employing (6.20) we get

$$\begin{aligned}\left\|\int_{x+1}^{\infty} dz \int_{\Gamma_+} d\lambda e^{\lambda(x-z)}T_\lambda h(z)\right\|_r &\\ &= \left\|\int_{x+1}^{\infty} dz(x-z)^{-1} \int_{\Gamma_+} d\lambda\, e^{\lambda(x-z)}\frac{\partial}{\partial\lambda}T_\lambda h(z)\right\|_r \\ &\leqq cc_0 e^{-\alpha N^2}.\end{aligned} \tag{6.21}$$

Using (6.19) we estimate

$$\left\|\int_{x}^{x+1} dx \int_{\Gamma_+} d\lambda e^{\lambda(x-z)}T_\lambda h(z)\right\|_r \leqq cc_0 e^{-\alpha N^2}. \tag{6.22}$$

Therefore

$$\left\|\int_{x}^{\infty} dz \int_{\Gamma_+} d\lambda e^{\lambda(x-z)}T_\lambda h(z)\right\|_r \leqq cc_0 e^{-\alpha N^2}. \tag{6.23}$$

(ii) Define new paths

$$\begin{aligned}\Sigma_+ &= \left\{\frac{v_1}{2}N^{\gamma-1} + x \pm 4ix, \quad x \geqq 0\right\},\\ \Sigma_- &= \left\{-\frac{v_1}{2}N^{\gamma-1} - x \pm 4ix, \quad x \geqq 0\right\}.\end{aligned} \tag{6.24}$$

By proposition 4.1 the contours Γ_+ and Γ_- can be deformed to Σ_+ and Σ_- without passing through singularities of $R_{N\lambda}$. Thus

$$\int_{\Gamma_\pm} e^{\lambda z}R_{N\lambda}d\lambda = \int_{\Sigma_\pm} e^{\lambda z}R_{N\lambda}d\lambda. \tag{6.25}$$

Next calculate

$$R_{N\lambda} = S_{N\lambda} - R_{N\lambda}HS_{N\lambda}, \tag{6.26}$$

$$\frac{\partial}{\partial\lambda}R_{N\lambda}HS_{N\lambda} = -R_{N\lambda}(\xi_1 - s)R_{N\lambda}HS_{N\lambda} + R_{N\lambda}HS_{N\lambda}(\xi_1 - s)S_{N\lambda}, \tag{6.27}$$

and estimate

$$\|R_{N\lambda}HS_{N\lambda}h\|_r \leqq (1+\lambda)^{-1/2}\|h\|_{r-2}, \tag{6.28}$$

$$\left\|\frac{\partial}{\partial\lambda}R_{N\lambda}HS_{N\lambda}h\right\|_r \leqq (1+\lambda)^{-1/2}\|h\|_{r-1-\gamma}. \tag{6.29}$$

Calculate the integral in three parts. By a contour integration,

$$\left\| \int_x^\infty dz \int_{\Sigma_\pm} d\lambda e^{\lambda(x-z)} S_{N\lambda} h(z) \right\|_r = \left\| \int_x^\infty dz (\xi_1 - s)^{-1} e^{-(\xi_1 - s)^{-1}\nu(x-z)} \chi_N h(z) \right\|_r$$

$$\leqq c_1 N^{-1} \int_{x+1}^\infty e^{-\nu_1 N^{\gamma-1}(x-z)} \max(e^{-\mu|z|^\beta}, e^{-\varepsilon|z|}) dz + \nu^{-1} N^{-\gamma} A(\varepsilon x), \tag{6.30}$$

$$\leqq CC_1 A(\varepsilon x),$$

using (6.2) and Lemmas 6.2 and 6.3. Next use (6.28) to obtain

$$\left\| \int_x^{x+1} dz \int_{\Sigma_+} d\lambda e^{\lambda(x-z)} R_{N\lambda} H S_{N\lambda} h(z) \right\|_r$$

$$\leqq c_1 \int_x^{x+1} dz \int_{(1/2)\nu_1 N^{\gamma-1}}^\infty d\lambda (1+\lambda)^{-1/2} e^{\lambda(x-z)} A(\varepsilon z) \leqq c c_1 A(\varepsilon x). \tag{6.31}$$

By integration by parts in λ, (6.6), (6.9), and (6.10), we can estimate

$$\left\| \int_{x+1}^\infty dz \int_{\Sigma_+} d\lambda e^{\lambda(x-z)} R_{N\lambda} H S_{N\lambda} h(z) \right\|_r$$

$$\leqq \left\| \int_{x+1}^\infty dz (x-z)^{-1} \int_{\Sigma_+} d\lambda e^{\lambda(x-z)} \frac{d}{d\lambda} (R_{N\lambda} H S_{N\lambda}) h(z) \right\|_r$$

$$\leqq c_1 \int_{x+1}^\infty dz (x-z)^{-1} \int_{(1/2)\nu_1 N^{\gamma-1}}^\infty e^{\lambda(x-z)} (1+\lambda)^{-1/2} A(\varepsilon z) d\lambda$$

$$\leqq c_1 A(\varepsilon x). \tag{6.32}$$

Combining (6.30), (6.31), and (6.32) and using (6.25) and (6.26) shows that

$$\left\| \int_x^\infty dz \int_{\Gamma_+} d\lambda e^{\lambda(x-z)} R_{N\lambda} h(z) \right\|_r \leqq c \cdot c_1 A(\varepsilon x). \tag{6.33}$$

Estimates similar to (6.17), (6.23), and (6.33) can also be obtained for the integrals over Γ_-. Combining these and using (6.14) results in (6.4). By setting $y = \varepsilon^{-1} x$ we can change Theorem 6.1 to:

Corollary 6.4. *The solution of*

$$(\xi_1 - s) \frac{\partial}{\partial x} w = \varepsilon^{-1} M w + \varepsilon^{-1} h, \tag{6.34}$$

with

$$\sup_x \| h(x) \|_{\alpha, r-\gamma} \leqq c_0, \tag{6.35}$$

$$\| h(x) \|_r \leqq c_1 A(x), \tag{6.36}$$

satisfies

$$\sup_x \| w(x) \|_{\alpha, r} \leqq c \cdot c_0, \tag{6.37}$$

$$\| w(x) \|_r \leqq c(c_0 + c_1) A(x). \tag{6.38}$$

Finally we also make estimates on the linearized version of equation (3.54) with asymptotic conditions (3.59).

Lemma 6.5. (i) *For any* $\tau_1 > 0$,

$$\int_0^x e^{-\tau_1(x-y)-\mu\varepsilon^{-\beta}y^\beta}dy \leqq c(xe^{-\tau_1 x} + e^{-\mu\varepsilon^{-\beta}x^\beta}); \tag{6.39}$$

(ii) *There is an* $X > 0$ *and* $\frac{1}{2}\tau_0 > \tau_1 > 0$, *such that*

$$\begin{aligned} -\tau + 2\gamma z_0(x) &> 2\tau_1, \quad \text{if } x > X, \\ &< -2\tau_1, \quad \text{if } x < -X. \end{aligned} \tag{6.40}$$

Theorem 6.6. *If*

$$|b(x)| \leqq c_0 A(x), \tag{6.41}$$

then the solution z_1 *of (3.54) and (3.59) satisfies*

$$|z_1(x)| \leqq c \cdot c_0 A(x). \tag{6.42}$$

7. Solution of the Nonlinear Equations

Using the preceding estimates on the linearized version of Eq. (3.55), we are ready to solve the full nonlinear equations (3.54) and (3.55) with the asymptotic conditions (3.59).

Theorem 7.1. *There is a solution of (3.54), (3.55), and (3.59) with*

$$|z_1(x)| \leqq cA(x), \tag{7.1}$$

$$\| w_1(x) \|_{\alpha,r} \leqq c, \tag{7.2}$$

$$\| w_1(x) \|_r \leqq cA(x), \tag{7.3}$$

for any r, $0 \leqq \alpha < \frac{1}{4}$ *and for* μ, β *as in (6.5).*

Once this is proved we have finished the construction of the shock profile and the proof of Theorem 2.1. First we make estimates on the inhomogeneities a and b in (3.54) and (3.55).

Lemma 7.2. *Suppose that*

$$|\tilde{z}_1(x)| \leqq c_0 A(x), \tag{7.4}$$

$$\| \tilde{w}_1(x) \|_{\alpha,r} \leqq c_1, \tag{7.5}$$

$$\| \tilde{w}_1(x) \|_r \leqq c_2 A(x). \tag{7.6}$$

If $\tilde{a}$ *and* $\tilde{b}$ *are defined by (3.56) and (3.57) with* $z_1 = \tilde{z}_1$ *and* $w_1 = \tilde{w}_1$ *then*

$$|\tilde{a}(x)| \leqq c(1 + \varepsilon c_0 + c_2 + \varepsilon(c_0 + c_2)^2)A(x), \tag{7.7}$$

$$\| \tilde{b}(x) \|_{\alpha,r-\gamma} \leqq c(1 + \varepsilon(c_0 + c_1) + \varepsilon^2(c_0 + c_1)^2), \tag{7.8}$$

$$\| \tilde{b}(x) \|_{r-\gamma} \leqq c(1 + \varepsilon(c_0 + c_2) + \varepsilon^2(c_0 + c_2)^2)A(x). \tag{7.9}$$

Lemma 7.3. *Suppose* $\tilde{z}_1$ *and* $\bar{z}_2$ *both satisfy (7.4) and* $\tilde{w}_1$ *and* $\bar{w}_1$ *both satisfy (7.5)* *and* (7.6). *Let* $(\tilde{a}, \tilde{b})$ *and* $(\bar{a}, \bar{b})$ *be defined by* (3.56) *and* (3.57) *using* $(\tilde{z}_1, \tilde{w}_1)$ *and* $(\bar{z}_1, \bar{w}_1)$ *respectively. Suppose also that*

$$|(\tilde{z}_1 - \bar{z}_1)(x)| \leqq d_0 A(x), \tag{7.10}$$

$$\|(\tilde{w}_1 - \bar{w}_1)(x)\|_{\alpha,r} \leqq d_1, \tag{7.11}$$

$$\|(\tilde{w}_1 - \bar{w}_1)(x)\|_r \leqq d_2 A(x). \tag{7.12}$$

Then

$$|(\tilde{a} - \bar{a})(x)| \leqq \{(\varepsilon d_0 + d_2) + \varepsilon(c_0 + c_2)(d_0 + d_2)\} A(x), \tag{7.13}$$

$$\|(\tilde{b} - \bar{b})(x)\|_{\alpha,r-\gamma} \leqq \varepsilon(d_0 - d_1) + \varepsilon^2(c_0 + c_1)(d_0 + d_1), \tag{7.14}$$

$$\|(\tilde{b} - \bar{b})(x)\|_{r-\gamma} \leqq \{\varepsilon(d_0 + d_2) + \varepsilon^2(c_0 + c_2)(d_0 + d_2)\} A(x). \tag{7.15}$$

These will be proved using the following estimates on z_0 and w_0:

$$|z_0(x)| \leqq e^{-(1/2)\tau_0|x|}, \quad x < 0, \tag{7.16}$$

$$|z_0(x) - z_\infty| \leqq e^{-(1/2)\tau_0|x|}, \quad x > 0, \tag{7.17}$$

$$\|w_0(x)\|_{\alpha,r} \leqq e^{-(1/2)\tau_0|x|}, \quad x < 0, \tag{7.18}$$

$$\|w_0(x) - w^\infty\|_{\alpha,r} \leqq e^{-(1/2)\tau_0|x|}, \quad x > 0, \tag{7.19}$$

and a nonlinear bound coming from Proposition 5.1 of [2].

Lemma 7.4. *If* $0 \leqq \alpha < \frac{1}{4}$,

$$\|\nu\Gamma(f,g)\|_{\alpha,r} \leqq c(\|f\|_r \|g\|_{\alpha,r+\gamma} + \|f\|_{\alpha+r,\gamma}\|g\|_r + \|f\|_{\alpha,r+\gamma-1}\|g\|_{\alpha,r+\gamma-1}). \tag{7.20}$$

Proof of Lemma 7.2. Denote

$$\begin{aligned}
\tilde{a}_1 &= -\tau' z_0 + \langle \mu_\varepsilon, z_0\psi_\varepsilon + w_0)\rangle - \langle \psi_\varepsilon, \nu\Gamma(2\phi_\varepsilon z_0 + \varepsilon w_0, w_0)\rangle, \\
\tilde{a}_2 &= \varepsilon\gamma \tilde{z}_1^2 + \langle \mu_\varepsilon, e\tilde{z}_1\psi_\varepsilon + \tilde{w}_1\rangle, \\
\tilde{a}_3 &= -\langle \mu_\varepsilon, \nu\Gamma(2\varepsilon\phi_\varepsilon\tilde{z}_1 + \varepsilon\tilde{w}_1, w_0 + \tilde{w}_1) + \nu\Gamma(2\phi_\varepsilon z_0 + \varepsilon w_0, \tilde{w}_1)\rangle.
\end{aligned} \tag{7.21}$$

By (7.16)–(7.19), (3.49), and (3.51), $|\tilde{a}_1(x)| \leqq ce^{-(1/2)\tau|x|}$. By (7.4) and (7.6), $|\tilde{a}_2(x)| \leqq (\varepsilon c_0^2 + \varepsilon c_0 + c_2)A(x)$. By (7.16)–(7.19), (7.4), (7.6), and (7.20), $|\tilde{a}_3(x)| \leqq c(\varepsilon c_0 + \varepsilon c_0 c_2 + \varepsilon c_2 + \varepsilon c_2^2)A(x)$. These estimates imply (7.7) since $\tilde{a} = \tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3$ and $\tau_1 < \frac{1}{2}\tau_0$. Denote

$$\begin{aligned}
\tilde{b}_1 &= -\varepsilon(\xi_1 - s)\frac{\partial}{\partial x}w_0 - Mw_0 - \varepsilon M_3 w_0 + \varepsilon z_0(I - P) \\
&\quad + (I - P)\nu\Gamma(z_0\phi_\varepsilon + \varepsilon w_0, z_0\phi_\varepsilon + \varepsilon w_0), \\
\tilde{b}_2 &= -\varepsilon M_3\tilde{w}_1 + \varepsilon^2\tilde{z}_1(I - P)\mu_\varepsilon, \\
\tilde{b}_3 &= (I - P)\{2\varepsilon\nu\Gamma(z_0\phi_\varepsilon + \varepsilon w_0, \tilde{z}_1\phi_\varepsilon + \tilde{w}_1) + \varepsilon^2\nu\Gamma(\tilde{z}_1\phi_\varepsilon + \tilde{w}_1, \tilde{z}_1\phi_\varepsilon \\
&\quad + \tilde{w}_1)\}.
\end{aligned} \tag{7.22}$$

By (7.16)–(7.19) and (3.50), $\|\tilde{b}_1(x)\|_{\alpha,r} \leqq ce^{-(1/2)\tau_0|x|}$. By (7.4)–(7.6),

$$\begin{aligned}
\|\tilde{b}_2(x)\|_{\alpha,r} &\leqq (\varepsilon c_0 + \varepsilon^2 c_1), \\
\|\tilde{b}_2(x)\|_r &\leqq (\varepsilon c_0 + \varepsilon^2 c_2)A(x).
\end{aligned} \tag{7.23}$$

By (7.16)–(7.19), (7.4)–(7.6), and (7.20),

$$\|\tilde{b}_3(x)\|_{\alpha,r} \leqq \{\varepsilon(c_0+c_1)+\varepsilon^2(c_0+c_1)^2\},$$
$$\|\tilde{b}_2(x)\|_{r-\gamma} \leqq \{\varepsilon(c_0+c_1)+\varepsilon^2(c_0+c_1)^2\}A(x). \tag{7.24}$$

Combining these yields (7.9).

Proof of Lemma 7.3. In notation like that in the previous proof, $\tilde{a}_1 - \bar{a}_1 = \tilde{b}_1 - \bar{b}_1 = 0$. This eliminates the term contributing the "1" on the right side of (7.7)–(7.9). The remaining terms are estimated as above to find (7.13)–(7.14).

Proof of Theorem 7.1. We are now ready to solve Eq. (3.54), (3.55), and (3.59) by iteration. Let $z_1^0 = w_1^0 = 0$, define a^n and b^n by (3.56) and (3.57) with z_1 and w_n replaced by z_1^n and w_1^n, and let z_1^{n+1} and w_1^{n+1} solve (3.54), (3.55), and (3.59) with a and b replaced by a^n and b^n. Then z_1^0 and w_1^0 satisfy (7.4)–(7.6) (for suitable c_0, c_1, c_2). The estimate (7.7)–(7.9) for a^0 and b^0 combine with Theorems 6.1 and 6.6 to find estimates on z_1^1 and w_1^1. By iterating the procedure we obtain uniform estimates on z_1^n and w_1^n. Choose C_0, C_1, and C^2 such that

$$c(1+\varepsilon C_0 + C_2 + \varepsilon(C_0+C_2)^2) \leqq C_0,$$
$$c(1+\varepsilon(C_0+C_1)+\varepsilon^2(C_0+C_1)^2) \leqq C_1,$$
$$c(1+\varepsilon(C_0+C_2)+\varepsilon^2(C_0+C_2)^2) \leqq C_2.$$

Then z_1^n and w_1^n satisfy (7.1)–(7.3) with $C = \max(C_0, C_1, C_3)$. Lemma 7.3 shows that $A(x)^{-1}|z_1^{n+1} - z_1^n|$, $\|w_1^{n+1}(x) - w_1^n(x)\|_{\alpha,r}$ and $A(x)^{-1}\|w_1^{n+1}(x) - w_1^n(x)\|_r$ are decreasing algebraically fast. Therefore z_1^n and w_1^n converge to z_1 and w_1 with bounds (7.1)–(7.3). By a standard argument, they are seen to be solutions of (3.54), (3.55), and (3.59).

To show the uniqueness of the solution we first write f as $f = (z_0 + \varepsilon z_1)\phi + \varepsilon(w_0 + w_1)$ as in Sect. 3. Then z_1 and w_1 solve (3.54), (3.55), (3.59). Assuming (2.26), we wish to show that $A(x)^{-1}|z_1(x)|$, $\|w_1(x)\|_{\alpha,r}$, and $A(x)^{-1}\|w_1(x)\|_r$ are bounded independent of ε. From (2.26) and Lemma (7.2) we find that $\|b(x)\|_{\alpha,r-\gamma}$ and $A(x)^{-1}\|b(x)\|_{r-\gamma}$ are bounded. Then Corollary 6.4 implies bounds on $\|w_1(x)\|_{\alpha,r}$ and $A(x)^{-1}\|w_1(x)\|_r$. Using Lemma 7.2 again we find bounds on $A(x)^{-1}|a(x)|$, and again Theorem 6.6 implies bound on $A(x)^{-1}|z_1(x)|$.

Next we suppose that f and $\bar{f}$ are two such solutions with $w_1 - \bar{w}_1$ and $z_1 - \bar{z}_1$ bounded in the above norms by a constant c. By Lemma 7.3 we find that $a - \bar{a}$, $b - \bar{b}$ are bounded by εc. Then by Corollary 6.4 and Theorem 6.6 we find that $z_1 - \bar{z}_1$ and $w_1 - \bar{w}_1$ are bounded by εc. Continuing this we find that $w_1 = \bar{w}_1$, $z_1 = \bar{z}_1$ and thus $f = \bar{f}$.

Appendix A. The Collision Operator

The nonlinear collision operator is

$$Q(f,g) = \tfrac{1}{2}\int_{R^3}\int_0^{2\pi}\int_0^{\pi/2} \{f(\xi_1')g(\xi') + f(\xi')g(\xi_1') - f(\xi_1)g(\xi)$$
$$- f(\xi)g(\xi_1)\}B(\theta, V)\,d\theta\,d\varepsilon\,d\xi_1, \tag{A.1}$$

in which

$$
\begin{aligned}
V &= \xi_1 - \xi, \\
\xi' &= \xi + \alpha(\alpha \cdot \mathbf{V}), \\
\xi_1' &= \xi_1 - \alpha(\alpha \cdot \mathbf{V}), \\
\alpha &= (\cos\theta, \sin\theta\cos\varepsilon, \sin\theta\sin\varepsilon),
\end{aligned}
\tag{A.2}
$$

and $B(\theta, \mathbf{V})$ is (the collision cross section)$\cdot V$. For an inverse power force $F(r) = \kappa r^{-3}$ with $3 < s$ and r the intermolecular distance, $B(\theta, \mathbf{V}) = V^{\gamma}\beta(\theta)$ with $\gamma = \dfrac{s-5}{s-1}$. In particular for inelastic collisions between spheres, $B(\theta, V) = V\cos\theta\sin\theta$. Define

$$
\begin{aligned}
\nu(\xi) &= 2\pi \int_{R^3} \int_0^{\pi/2} B(\theta, \boldsymbol{\eta} - \xi)\omega(\boldsymbol{\eta})d\theta d\boldsymbol{\eta}, \\
\omega(\xi) &= (2\pi)^{-1/2} e^{-\xi^2/2}.
\end{aligned}
\tag{A.3}
$$

We shall consider only hard potentials with an angular cutoff in the sense of Grad [11], i.e. we assume that

$$
\begin{aligned}
\nu_1 \cdot (1+\xi)^{\gamma} \leqq \nu(\xi) \leqq \nu_2 \cdot (1+\xi)^{\gamma}, \\
B(\theta, V) \leqq cV^{\gamma}\sin\theta\cos\theta|,
\end{aligned}
\tag{A.4}
$$

in which ν_1, ν_2 and c are positive constants and $0 \leqq \gamma \leqq 1$ and that B is continuous in V. Power law forces do not satisfy these constraints; some modification is required to eliminate grazing collisions with θ small.

The linearized collision operator is $Lf = -2\omega^{-1/2}Q(\omega, \omega^{1/2}f)$. Using (A.2) and positivity and symmetry properties of B, one can show that L is self-adjoint and non-negative, with $N(L) = \operatorname{span}\{\chi_i, i = 0, \dots, 4\}$ [11]. It can be represented as $L = \nu(\xi) - K$ with

$$
\begin{aligned}
Kf(\xi) &= \int_{R^3} k(\xi, \boldsymbol{\eta}) f(\boldsymbol{\eta}) d\boldsymbol{\eta}, \\
k &= -k_1 + k
\end{aligned}
\tag{A.5}
$$

$$
k_1(\xi, \boldsymbol{\eta}) = 2\pi\omega(\xi)^{1/2}\omega(\boldsymbol{\eta})^{1/2} \int_0^{\pi/2} B(\theta, \mathbf{V}) d\theta,
\tag{A.6}
$$

$$
\begin{aligned}
k_2(\xi, \boldsymbol{\eta}) = 2(2\pi)^{-3/2} v^{-2} \exp\{-\tfrac{1}{8}v^2 - \tfrac{1}{2}\zeta_1^2\} \\
\cdot \int \exp\{-\tfrac{1}{2}|\mathbf{w} + \xi_2|^2\} q(\mathbf{v}, \mathbf{w}) d\mathbf{w}
\end{aligned}
\tag{A.7}
$$

in which

$$
\begin{aligned}
\mathbf{v} &= \boldsymbol{\eta} - \xi = \alpha(\alpha \cdot \mathbf{V}), \\
\mathbf{w} &= \mathbf{V} - \alpha(\alpha \cdot \mathbf{V}), \\
\zeta_1 &= \mathbf{v}(\mathbf{v} \cdot \tfrac{1}{2}(\xi + \boldsymbol{\eta})), \\
\zeta_2 &= \tfrac{1}{2}(\xi + \boldsymbol{\eta}) - \zeta_1.
\end{aligned}
\tag{A.8}
$$

Note that $\mathbf{w}$ is perpendicular to $\mathbf{v}$ and the integral in (A.7) is over the two-dimensional plane with $\mathbf{v}$ held constant. Also we define

$$q(\mathbf{v},\mathbf{w})=(2|\sin\theta|^{-1}[B(\theta,\mathbf{V})+B(\tfrac{\pi}{2}-\theta,\mathbf{V})\})\leqq cv(v^2+w^2)^{-(1-\gamma)/2}. \tag{A.9}$$

The bounds (4.10) for H replaced by K and (4.15) for $\bar{k}$ replaced by k were derived in [2] along with the bound

$$\|Kh\|_{3/2-\gamma}\leqq c\|f\|_{L^2}. \tag{A.10}$$

These can be used as in [11] to show the following:

Lemma A.1. *If* $\gamma\leqq 1$, $H=e^{\alpha\xi^2}v(\xi)^{-1}Ke^{-\alpha\xi^2}$ *is compact as an operator on* L^2.

Lemma A.2. *Let* $h\in G_{\alpha,r}$ *with* $\langle\chi_i,h\rangle=0$, $i=0,\dots,4$, *then* $Lf=h$ *has a solution with* $f\in G_{\alpha,r+\gamma}$.

Appendix B. The Summational Invariants

The summational invariants $\chi_0,\dots,\chi_4$ defined in (2.9) form an orthonormal basis for the null space of L. The sound speed c_0 is found as a root of $\det(\langle\chi_i,(\xi_1-c_0)\chi_j\rangle)=-c_0^3(c_0-\frac{5}{3})$, i.e. $c_0=\sqrt{5/3}$. One could also use $c_0=-\sqrt{5/3}$; the roots $c_0=0$ correspond to contact discontinuities rather than shocks. The function $\phi_0'=\Sigma\alpha_i\chi_i$ is found through a null vector (α_i) of the matrix $\langle\chi_i,(\xi_1-c_0)\chi_j\rangle$, $(i,j=0,\dots,4)$ which is $\alpha_0=1$, $\alpha_1=c_0$, $\alpha_2=\alpha_3=0$, $\alpha_4=\sqrt{2/3}$ as in (3.11).

Next we solve (3.14), (3.17), and (3.18). Rewrite $\sum_{i=0}^{4}\beta_i\chi_i=\beta'\phi_0'+\sum_{i=0}^{3}\beta_i'\chi_i$. Then (3.14) is

$$\left\langle(\xi_1-c_0)\chi_i,\sum_{j=0}^{3}\beta_j'\chi_j\right\rangle=-\langle\chi_i,\phi_0'\rangle-\langle(\xi_1-c_0)\chi_i,\phi_1'\rangle,\quad i=0,\dots,3. \tag{B.1}$$

Since ϕ_1 is to be independent of ε, (3.18) implies that $\langle(\xi_1-c_0)\phi_1,\phi_1\rangle=0$. This is just a linear equation for β' with coefficient $2\langle(\xi_1-c_0)\phi_0,\phi_1'\rangle=2\tau^{-1}\langle L\phi_1',\phi_1'\rangle\neq 0$.

Also we calculate $\det(\langle(\xi_1-c_0)\chi_i,\chi_j\rangle)$ $(i,j=-1,\dots,4)$ with $\chi_{-1}=\phi_1$. From (3.21) and (3.14) we have $\langle(\xi_1-c_0)\chi_i,\phi_1\rangle=-\langle\chi_i,\phi_0\rangle$. This makes it possible to calculate the determinant to be $c_0^2\alpha^2(-c_0^4-3c_0^2+\frac{8}{3}c_0-\frac{2}{3})\neq 0$. Since $\det(\langle\chi_i,(\xi_1-c_0)\chi_j\rangle)(i,j=0,\dots,3)=c_0^2(c_0^2-1)\neq 0$, (B.1) has a unique solution $\beta_0',\dots,\beta_3'$. Look for θ_ε and $\theta_\varepsilon=\sum_{i=0}^{3}\gamma_i\chi_i+\tilde{\theta}$ will be choosen to satisfy $\langle(\xi_1-s)\chi_i,\tilde{\theta}\rangle=0,\dots,3$. Then (3.17) for $i=0,\dots,3$ becomes

$$\left\langle(\xi_1-s)\chi_i,\sum_{j=0}^{3}\gamma_j\chi_j\right\rangle=-\langle\chi_i,\phi_1'\rangle,\qquad i=0,\dots,3. \tag{B.2}$$

Since $\det(\langle\chi_i,(\xi_1-s)\chi_i\rangle)$ is bounded uniformly in ε near 0, the γ_j's are determined uniquely and are uniformly bounded. For the last equation of (3.17) we can replace

χ_4 by ϕ_0. This and (3.18) give us two equations for β' and $\tilde{\theta}$:

$$\langle \phi_0, \phi_1 \rangle + \langle (\xi_1 - c_0)\tilde{\theta}_0, \tilde{\theta} \rangle + O(\varepsilon) = 0, \tag{B.3}$$

$$2\langle \phi_0, \phi_1 \rangle + \langle (\xi_1 - c_0)\phi_1, \phi_1 \rangle + 2\langle (\xi_1 - c_0)\phi_0, \tilde{\theta} \rangle + O(\varepsilon) = 0. \tag{B.4}$$

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