Periodic solutions for three sedimenting spheres

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Exact periodic solutions are found for the relative motion of three spheres sedimenting in a Stokes fluid. Nearby solutions are found to be nearly periodic for a long time. Existence of the exact periodic solutions is proved using the point-particle approximation and symmetry properties of Stokes equations. Numerical simulations for finite-sized particles are performed using a method of multipole expansions.

I. INTRODUCTION

Sedimentation of solid spheres in a viscous liquid is described by Stokes equations if the Reynolds number based on sphere radius is sufficiently small. Since inertial forces are neglected in the approximation, two spheres of equal mass and size will fall without changing their relative position and at a speed larger than that of a single particle. For three such spheres, the simplest scenario for their motion[1–3] is that a pair of particles starting far above the third will catch up with it. After the three interact, a pair of particles (not necessarily the same two) will eventually pull away and fall faster than the third.

The point of this paper is to describe a new class of solutions for which the relative motion of the three spheres is periodic in time. For such periodic solutions, the three particles will always remain close, and their motion cannot be described as a collision between a pair of particles and a third particle.

These periodic solutions can be described as follows: The three particles start off in a horizontal plane with two of them closer together than to the third. The two initially fall faster than the third, but are spread apart because of the fluid flow caused by the third particle. Eventually they are pushed far enough apart that the third particle falls faster than the pair and catches up with them. Now the reverse happens. The pair is moving more slowly than the third particle, but the flow above the third particle pulls them closer together. Eventually they start falling faster than the third particle and catch up with it. If enough symmetry is imposed initially, the relative positions in the final configuration are found to be exactly the same as those in the initial configuration, so that the motion is periodic.

These solutions will be shown to occupy a four-dimensional subset of the six-dimensional phase space for the relative positions of three particles. Moreover, nearby solutions are found to be nearly periodic for a surprisingly long time, so that the periodic solutions are robust (if not actually stable). Although these periodic solutions are rather special, they can be expected to be useful for understanding the motion of larger numbers of particles and as numerical test problems.

Recently Durlofsky, Brady, and Bossis[4] constructed a periodic solution for a configuration of four particles. An analytic treatment of flow past three particles on the vertices of a horizontal equilateral triangle was presented by Kim.[5]

The construction of the periodic solutions employs symmetry properties of Stokes equations detailed in Sec. IV, and periodic solutions for three point particles, which were first found by Hocking[3] and are described in Secs. II and III. Robustness of the periodic solutions is demonstrated at the end of Sec. III. Numerical computations for three sedimenting spheres are presented in Sec. V showing periodicity for symmetric configurations and near periodicity for asymmetric configurations. Some implications of these results are outlined in the concluding Sec. VI.

II. THE POINT-PARTICLE APPROXIMATION

The motion of spheres in a Stokes fluid may be approximated by that of point particles. Each point particle applies a point force $mg\mathbf{e}_1$ to the fluid, in which m is the particle mass, g is the gravitational constant, and $\mathbf{e}_1$ is the unit vector pointing downward. The fluid velocity at $\mathbf{y}$ resulting from such a single point particle at $\mathbf{y}-\mathbf{x}$ is given by the Stokeslet $\kappa\mathbf{U}^{(1)}(\mathbf{x})$ in which $\kappa = mg/8\pi\mu$ and

$$\mathbf{U}(\mathbf{x}) = [\mathbf{e}_1/|\mathbf{x}| + (\mathbf{e}_1\cdot\mathbf{x}\mathbf{x})/|\mathbf{x}|^3]. \tag{1}$$

According to Faxen's first law,[6,7] the velocity $\mathbf{v}$ of a point particle, sitting in a flow $\mathbf{u}$ and feeling a force $\mathbf{F}$, is given by $\mathbf{v} = \mathbf{u} + \mathbf{v}_{\mathrm{St}}$ in which $\mathbf{v}_{\mathrm{St}}$ is the Stokes velocity corresponding to the force $\mathbf{F}$. In order that $\mathbf{v}_{\mathrm{St}} = (6\pi\mu a)^{-1}\mathbf{F}$ be finite, the point particle must be considered to have an effective radius a.

For three point particles, each particle feels an ambient flow $\mathbf{u}$ from the sum of the Stokeslets generated by the other two particles. Thus if the particle positions are $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$, their velocities are, respectively,

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$$\dot{\mathbf{x}}^{(1)} = \mathbf{v}_{St} + \kappa \mathbf{U}(\mathbf{x}^{(1)} - \mathbf{x}^{(2)}) + \kappa \mathbf{U}(\mathbf{x}^{(1)} - \mathbf{x}^{(3)}),$$
$$\dot{\mathbf{x}}^{(2)} = \mathbf{v}_{St} + \kappa \mathbf{U}(\mathbf{x}^{(2)} - \mathbf{x}^{(1)}) + \kappa \mathbf{U}(\mathbf{x}^{(2)} - \mathbf{x}^{(3)}), \quad (2)$$
$$\dot{\mathbf{x}}^{(3)} = \mathbf{v}_{St} + \kappa \mathbf{U}(\mathbf{x}^{(3)} - \mathbf{x}^{(1)}) + \kappa \mathbf{U}(\mathbf{x}^{(3)} - \mathbf{x}^{(2)}).$$

Note that $\mathbf{U}(\mathbf{x}) = \mathbf{U}(-\mathbf{x})$. Equations (2) are simplified by translating to the frame centered at $\mathbf{x}^{(1)}$ and rescaling time. Let

$$\mathbf{y} = \mathbf{x}^{(2)} - \mathbf{x}^{(1)}, \quad \mathbf{z} = \mathbf{x}^{(3)} - \mathbf{x}^{(1)}, \quad t = \kappa t',$$

in which t' is the original time variable. Equations for $\mathbf{y}$ and $\mathbf{z}$ are

$$\dot{\mathbf{y}} = \mathbf{U}(\mathbf{y} - \mathbf{z}) - \mathbf{U}(\mathbf{z}),$$
$$\dot{\mathbf{z}} = \mathbf{U}(\mathbf{y} - \mathbf{z}) - \mathbf{U}(\mathbf{y}). \quad (3)$$

This six-dimensional system of ordinary differential equations (ODE's) forms the basis for this section as well as Secs. III and IV.

As observed by Hocking,[3] the system (3) has a simple conserved quantity given by

$$A = -\tfrac{1}{2}\mathbf{e}_1 \cdot (\mathbf{y} \times \mathbf{z}) = -\tfrac{1}{2}(y_2 z_3 - y_3 z_2), \quad (4)$$

which can be interpreted as the area of the horizontal projection of the triangle $(\mathbf{0},\mathbf{y},\mathbf{z})$. A simple calculation shows that

$$\dot{A} = 0. \quad (5)$$

The equilibrium solutions of (3) are horizontal equilateral triangles, i.e.,

$$\mathbf{y} = (0, y_2, y_3), \quad \mathbf{z} = (0, z_2, z_3),$$
$$|\mathbf{y}| = |\mathbf{z}| = |\mathbf{y} - \mathbf{z}|. \quad (6)$$

Linearization of (3) around the equilibrium solution (6) results in a linear system for which the eigenvalues are 0, 0, $\pm i\sqrt{\tfrac{3}{2}}|\mathbf{y}|^{-2}$, $\pm i\sqrt{\tfrac{3}{2}}|\mathbf{y}|^{-2}$. Thus there are two linearly degenerate modes and four periodic modes of period $T = 2\pi\sqrt{\tfrac{2}{3}}|\mathbf{y}|^2$ in the scaled time, or $T = 2\pi\sqrt{\tfrac{2}{3}}\kappa^{-1}|\mathbf{y}|^2$ in the original time.

III. PERIODIC MOTION FOR POINT PARTICLES

Periodic solutions for the point-particle equations (3) will be found by restricting to symmetric solutions. Assume that

$$\mathbf{y} = (y_1, y_2, y_3), \quad \mathbf{z} = (y_1, y_2, -y_3), \quad (7)$$

i.e., that $(\mathbf{0},\mathbf{y},\mathbf{z})$ forms an isosceles triangle with a horizontal base [any horizontal rotation of (7) would also work]. This symmetry is preserved by Eq. (3). Assume, without loss of generality, that $y_2 > 0$, $y_3 > 0$; since $A = y_2 y_3$ is constant in time, y_2 and y_3 will remain positive. Under these assumptions, the system (3) reduces to

$$\dot{y}_1 = (2A)^{-1} y_2 - (y_1^2 + y_2^2 + A^2 y_2^{-2})^{-1/2}$$
$$- y_1^2 (y_1^2 + y_2^2 + A^2 y_2^{-2})^{-3/2}, \quad (8)$$
$$\dot{y}_2 = -y_1 y_2 (y_1^2 + y_2^2 + A^2 y_2^{-2})^{-3/2}.$$

This second-order, autonomous ODE has an equilibrium point at $y_1 = 0$, $y_2 = (3A^2)^{1/4}$, corresponding to a horizontal equilateral triangle and around which the linearized flow is periodic. Experimental observations of this equilibrium are well known.[8] Because the system is symmetric about $y_2 = 0$, the full nonlinear flow for (8) is also periodic near this point. These periodic solutions are separated by a separatrix from the remainder of solutions which approach $y_1 = \pm\infty$ as $t \to \pm\infty$. The latter, which we call escaping solutions, correspond to a pair of particles catching up to and passing a third particle. The full phase plane for (8) is drawn in Fig. 1.

Numerical computations of a periodic solution for the full system (3) under the symmetry assumption (7) are sketched in Fig. 2. Note that of the six-dimensional phase space for Eqs. (3), the periodic solutions form a four-dimensional bounded subset, corresponding to the choice of y_1, y_2, y_3 and a horizontal rotation. Even under strong perturbations of the symmetric solutions, nearby solutions are nearly, but not exactly, periodic. Figures 3 and 4 show the evaluation of a typical asymmetric solution over the time ranges $0 \leqslant t \leqslant 30$ and $1800 \leqslant t \leqslant 1830$. The solution is nearly, but not exactly, periodic; it is remarkable that even after more than 100 "periods" the solution is still nearly periodic. This shows the periodic solutions to be very robust.

IV. PERIODIC MOTION FOR SPHERES

Using the periodic solutions for three point particles and the symmetry of Stokes equations, we shall deduce the existence of exact, periodic solutions for three spheres sedimenting in a Stokes fluid.

Stokes equations, for the motion of three spheres with a radius a and centers $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$ are the following: For $|\mathbf{x} - \mathbf{x}^{(i)}| > a$ $(i = 1,2,3)$,

$$\mu \nabla^2 \mathbf{u} - \nabla_p = 0, \quad (9)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (10)$$

with boundary conditions

$$\mathbf{u} = \dot{\mathbf{x}}^{(i)} - (\mathbf{x} - \mathbf{x}^{(i)}) \times \boldsymbol{\tau}^{(i)} \quad \text{on } |\mathbf{x} - \mathbf{x}^{(i)}| = a, \quad (11)$$

in which $\mathbf{v}^{(i)} = \dot{\mathbf{x}}^{(i)}$ is the particle velocity and $\boldsymbol{\tau}^{(i)}$ is the angular velocity. Both $\mathbf{v}^{(i)}$ and $\boldsymbol{\tau}^{(i)}$ are $\mathbf{x}$-independent vectors which are to be determined as part of the problem. In addition, there are equations of balance of force and torque on each sphere. For $i = 1,2,3$,

$$\int_{|\mathbf{x} - \mathbf{x}^{(i)}|} (\boldsymbol{\sigma} \cdot \mathbf{n})\, ds = -\mathbf{F}, \quad (12)$$

$$\int_{|\mathbf{x} - \mathbf{x}^{(i)}|} (\boldsymbol{\sigma} \cdot \mathbf{n}) \times \mathbf{n}\, ds = 0, \quad (13)$$

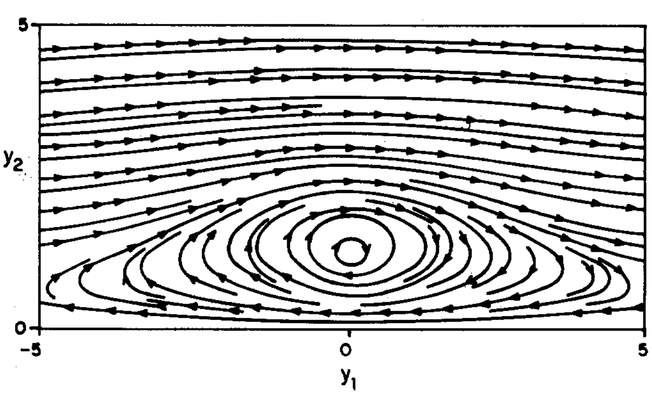

FIG. 1. Phase plane portrait for the time variation of the two horizontal components y_1, y_2 for a symmetric configuration of three point particles. This result is computed from Eq. (8) with $A = 1$.

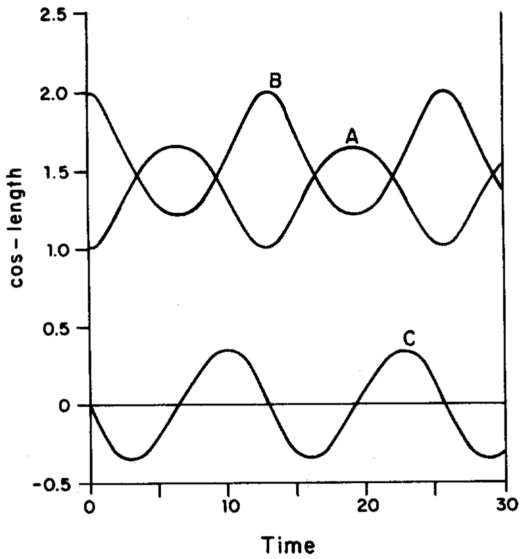

FIG. 2. Periodic solution for three point particles. Initial data: $\mathbf{y}(0) = (0,1,1)$, $\mathbf{z}(0) = (0,1,-1)$. Notation: A = length of midline $= |\mathbf{y}+\mathbf{z}|/2$, B = length of base $= |\mathbf{y}-\mathbf{z}|$, C = cosine of angle between midline and vertical $= (y_1+z_1)/|\mathbf{y}+\mathbf{z}|$, D = cosine of angle between base and vertical $= (y_1-z_1)/|\mathbf{y}-\mathbf{z}|$, E = cosine of angle between base and midline. For an isosceles triangle with a horizontal base, as in this case, $D=E=0$.

in which $\mathbf{F} = mg\mathbf{e}_1$ is a given force and $\boldsymbol{\sigma}$ is the stress tensor, $\sigma_{ij} = -p\delta_{ij} + \mu(u_{i,j} + u_{j,i})$. Equations (9)–(13) could be rewritten as a system of six ODE's for the relative positions $\mathbf{y} = \mathbf{x}^{(2)} - \mathbf{x}^{(1)}$ and $\mathbf{z} = \mathbf{x}^{(3)} - \mathbf{x}^{(1)}$, analogous to (3). Note that the solution of (9)–(13) is unique once the initial data $\mathbf{x}^{(i)}(t=0)$ $(i=1,2,3)$ is given.

If the three particles lie originally in a horizontal plane $x_1 = 0$, then their subsequent motion resulting from gravity is the mirror image about $x_1 = 0$ of the motion if the direction of gravity was reversed. Reversal of the direction of gravity is the same as reversal of time (since inertia is ignored). So an equivalent statement is that the motion forward in time is the mirror image of the motion backward in time. This can be stated by the following.

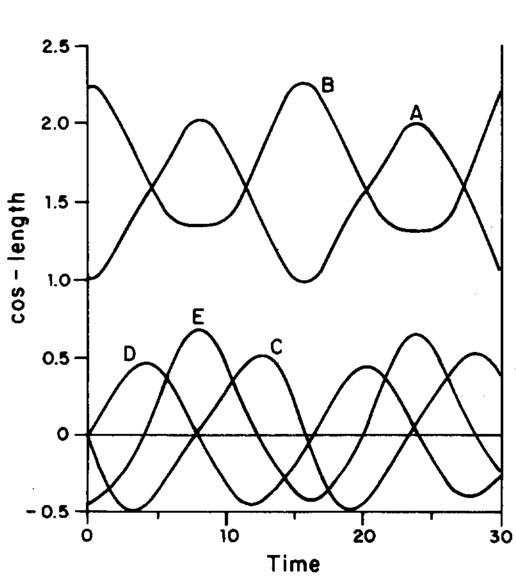

FIG. 3. A nearly periodic solution for three point particles starting from asymmetric initial data: $\mathbf{y}(0) = (0.0,0.5,1.0)$, $\mathbf{z}(0) = (0.0,1.5,-1.0)$.

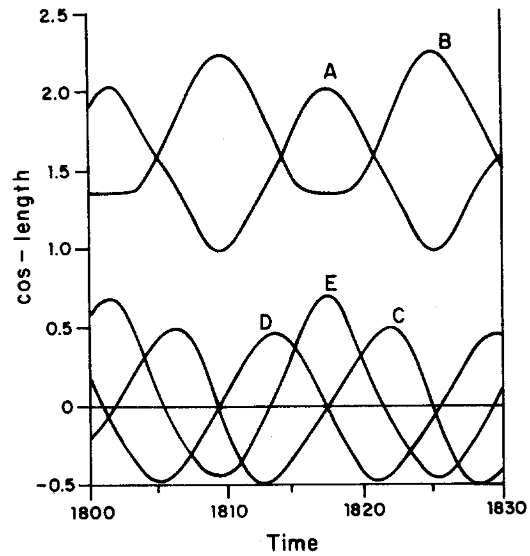

FIG. 4. Same as in Fig. 3, but after more than 100 periods.

Symmetry property (time reversal): If $x_1^{(i)}(t=0) = 0$ for $i = 1,2,3$, then $(x_1^{(i)},x_2^{(i)},x_3^{(i)})(t) = (-x_1^{(i)},x_2^{(i)},x_3^{(i)})(-t)$.

The same symmetry is true for the relative positions $\mathbf{y} = \mathbf{x}^{(2)} - \mathbf{x}^{(1)}$ and $\mathbf{z} = \mathbf{x}^{(3)} - \mathbf{x}^{(1)}$.

Symmetry property (time reversal for relative positions): If

$$y_1(t=0) = z_1(t=0) = 0,$$

then

$$(y_1,y_2,y_3)(t) = (-y_1,y_2,y_3)(-t)$$

and

$$(z_1,z_2,z_3)(t) = (-z_1,z_2,z_3)(-t).$$

A periodicity property can be inferred from this symmetry property. Suppose that the three particles lie in a horizontal plane at $t=0$ and in a (possibly different) horizontal plane at $t = T/2$. Time reversal and reflection about the horizontal plane at $t=0$ leads to horizontal configurations at times $t = T/2$ and $t = -T/2$. At these two times the configurations may lie in different horizontal planes (different vertical coordinates) but their horizontal coordinates are exactly the same, i.e.,

$$x_1^{(1)}(T/2) = x_1^{(2)}(T/2) = x_1^{(3)}(T/2),$$
$$x_1^{(1)}(-T/2) = x_1^{(2)}(-T/2) = x_1^{(3)}(-T/2),$$
$$x_2^{(i)}(T/2) = x_2^{(i)}(-T/2),$$
$$x_3^{(i)}(T/2) = x_3^{(i)}(-T/2),$$

but

$$x_1^{(1)}(T/2) \neq x_1^{(1)}(-T/2).$$

Thus the relative positions $\mathbf{y}$ and $\mathbf{z}$ are the same at these two times, i.e.,

$$y_1(T/2) = z_1(T/2) = y_1(-T/2) = z_1(-T/2) = 0,$$
$$y_2(T/2) = y_2(-T/2), \quad y_3(T/2) = y_3(-T/2),$$
$$z_2(T/2) = z_2(-T/2), \quad z_3(T/2) = z_3(-T/2).$$

This is summarized by the following.

Periodicity property: If $x_1^{(1)}(0) = x_1^{(2)}(0) = x_1^{(3)}(0)$ and $x_1^{(1)}(T/2) = x_1^{(2)}(T/2) = x_1^{(3)}(T/2)$, then the relative motion of the three spheres is T periodic, i.e.,

$$\mathbf{x}^{(i)}(t+T) - \mathbf{x}^{(1)}(t+T) = \mathbf{x}^{(i)}(t) - \mathbf{x}^{(1)}(t), \quad i = 2,3. \tag{14}$$

From this periodicity property, we derive our main result for periodic solutions of the three-sphere problem.

Existence of periodic solutions: Let $\mathbf{x}^{(i)}(t=0), i = 1,2,3$, lie on an isosceles triangle with horizontal base, i.e., $x_1^{(2)}(0) = x_1^{(3)}(0)$, $|\mathbf{x}^{(1)}(0)| = |\mathbf{x}^{(3)}(0) - \mathbf{x}^{(1)}(0)|$. Suppose that motion of point particles with this same initial data is periodic and that the radius a of the spheres is sufficiently small compared with their separation $\min(|\mathbf{x}^{(2)} - \mathbf{x}^{(1)}|, |\mathbf{x}^{(2)} - \mathbf{x}^{(3)}|)$. Then the relative motion of the spheres is exactly periodic, i.e., for some T,

$$\mathbf{x}^{(i)}(t+T) - \mathbf{x}^{(1)}(t+T) = \mathbf{x}^{(i)}(t) - \mathbf{x}^{(1)}(t). \tag{15}$$

To demonstrate this result, first note that by the symmetry of Stokes equations with respect to reflection in a vertical plane (i.e., with a horizontal normal vector), the sphere centers $\mathbf{x}^{(i)}$ will always be on an isosceles triangle with horizontal base. In particular, the relative motion of the three spheres is completely determined by the relative position of one of them $\mathbf{x}^{(2)} - \mathbf{x}^{(1)}$. Since the spheres' radii are small, their relative motion is well approximated by that of point particles, which is periodic. Although this does not yet imply that the relative motion of the spheres is periodic, it does imply that $x_1^{(1)}(t) - x_1^{(1)}(t) = 0$ at two times $t = t_1$ and $t = t_1 + T/2$. Since the spheres are on an isosceles triangle with horizontal base, $x_1^{(3)}(t) - x_1^{(1)}(t) = 0$ for $t = t_1$ and $t = t_1 + T/2$. Periodicity of the relative motion of the spheres then follows from the periodicity property.

Note that for Stokes equations (9)–(13) with periodic boundary conditions, the symmetry properties are still valid. Invariance with respect to reflection in a vertical plane is retained only if the vertical plane is parallel to a side of the periodicity cell. Therefore the existence of periodic solutions is still valid if the base of the isosceles triangle is parallel to a side of the periodicity cell.

Note also that since the point-particle approximation is valid, asymmetric solutions near the periodic orbits will be nearly periodic for a long time. Therefore the periodic solutions for three spheres are also robust as shown in Sec. V.

V. NUMERICAL COMPUTATIONS FOR SPHERES

Numerical computations for three sedimenting spheres were performed using a method developed by one of the authors (Sangani).[9] In this method, the velocity field is expanded in terms of various derivatives of a fundamental singular solution of the Stokes equation with the singularities situated at the center of each of the particles. The resulting field is then expanded in Legendre polynomials around each particle, and the strengths of singularities are determined from the no-slip boundary condition on the particles by evaluating the coefficients of Legendre polynomials of order less than or equal to N_p. The translational and rotational velocities of the particles thus computed are checked for convergence by repeating the calculations at higher N_p.

TABLE I. Initial vertical velocity for the sphere at origin at $t=0$ for $a=0.7$ (nearly touching) as a function of N_p = maximal order for Legendre polynomials. Horizontal velocity ≈ 0. Initial configuration was (0,0,0), (0,1,1), (0,1, − 1) (symmetric).

N_p	u_1
1	− 0.244
2	− 0.2256
3	− 0.2352
4	− 0.2110
5	− 0.2285
6	− 0.2110

First the results for symmetric configurations are presented. Initially let the particle centers be located at the points (0,0,0), (0,1,1), (0,1, − 1) as in the symmetric point-particle calculation in Sec. III. The maximum particle radius a for such a configuration is $a_{\max} = \sqrt{2}/2 = 0.707\ldots$.

Convergence of the numerical method as $N_p \to \infty$ is demonstrated in Tables I and II. Table I presents the (relative) vertical velocity u_1 of the spheres at $t=0$ for $a=0.7$ as a function of N_p. The value of u_1 is seen to vary by only 10% as N_p is increased from 2 to 6, even for these nearly touching spheres with $a = 0.7$. Table II presents the period T for the relative motion of the three sphere for $a = 0.6$, as a function of N_p and the time step Δt, showing good convergence.

These numerical computations for this symmetric configuration show the discretized motion to be very nearly periodic, verifying the periodicity of the actual continuous motion. The resulting period as a function of the particle radius is presented in Table III. The period is seen to increase as the radius increases. Also the point-particle approximation of Sec. III, which gave the half-period to be 6.5, appears to give an excellent estimation of the half-period for the finite size particles up to $a = 0.3$. The x_1 and x_2 coordinates of the particle (initially at the origin) after a time $T/2$ are also given in Table III. As a approaches zero, the x_1 coordinate of the particle after the half-period approaches infinity, and it is interesting to note that even when $a = 0.6$, the particles travel roughly 100 times their diameter in the direction of gravity before they complete one period. Stokes approximation will remain valid for such a long time only if the Reynolds number is extremely small.

For the asymmetric initial configuration (0,0,0), (0,0.5,1.0), (0,1.5, − 1.0), numerical computations for $a = 0.25$ and $a = 0.4$ give nearly periodic trajectories for a long time, just as for the point particles. The time dependence of the trajectory for $a = 0.4$ is displayed in Fig. 5.

TABLE II. Half-period as a function of N_p and time step Δt. Same initial configuration as Table I.

N_p	$\Delta t = 0.1$ ($T/2$)	$\Delta t = 0.2$ ($T/2$)
2	8	⋯
3	10	⋯
4	9.5	9.5
5	9.4	⋯

TABLE III. Half-period and the coordinates $x_i^{(1)}$ of the particle initially at the origin after the half-period ($x_3^{(1)} = 0$ for all times), as a function of particle radius a. Symmetric initial data as in Table I. Time step was $\Delta t = 0.1$.

a	$T/2$	$x_1^{(1)}(T/2)$	$x_2^{(1)}(T/2)$	N_p
0.1	6.5	95.3	− 1.335	2
0.3	6.65	38.5	− 1.33	3
0.5	7.9	32.2	− 1.29	3
0.6	9.5	34.5	− 1.31	4
0.65	13.6	47.6	− 1.56	4

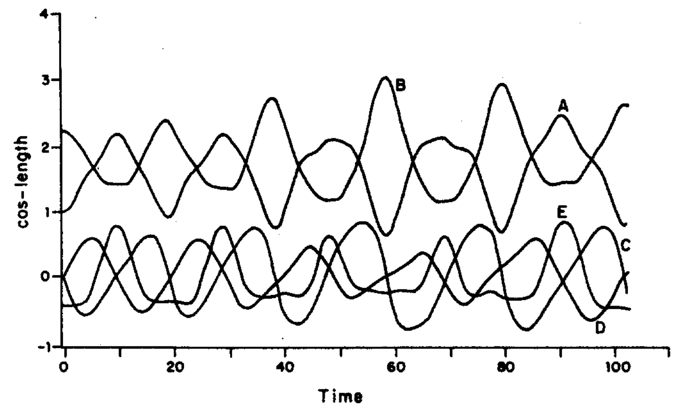

FIG. 5. A nearly periodic solution for three spheres with $a = 0.4$ starting from asymmetric initial data $\mathbf{y}(0) = (0,0.5,1.0)$, $\mathbf{z}(0) = (0,1.5,-1.0)$ as in Fig. 3, and with numerical parameters $N_p = 3$, $\Delta t = 0.05$.

VI. CONCLUSION

The periodic solutions constructed here are surprisingly robust. This suggests that they may be more generally useful for understanding the motion of three or more particles and also as a numerical test problem.

A possible application of these solutions is to a method for determination of the physically correct two-particle distribution function for sedimenting particles, which would be useful for many purposes such as analysis of hydrodynamic dispersion during sedimentation. A possible method for determination of the two-particle distribution function is a cluster expansion at low densities. The two-particle contribution to this expansion is degenerate since any two particles move without changing their relative position. Inclusion of the three-particle effect would involve these periodic and nearly periodic solutions. However, it is not clear that the three particle interactions are more important than the cumulative many-particle interactions.

After the conserved quantity A and rotational symmetry have been accounted for, the point-particle equations (3) are a four-dimensional dynamical system. As such, it would be of some interest to find almost periodic and chaotic solutions.

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