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Evaluation of a Function at Infinity from Its Power Series

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A new procedure for evaluating a function at infinity from its power series is analyzed. The limit in the procedure is shown to converge to the smallest singularity of the inverse function, which may or may not be the desired value.

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A new procedure for evaluating a function at infinity from its power series was formulated and successfully applied to quantum field theory and boundary-layer problems by Bender and co-workers.[1-7] In a number of interesting problems this procedure converges to the correct answer; in other problems it approaches the correct answer before diverging away. This led to a modification of the original method,[3] which seems to have improved convergence properties. Rivers[8,9] found that these methods could converge to an incorrect value and suggested a necessary condition for obtaining the correct limit. In this paper we show that the original method converges, at least in limit infimum, to the smallest singular point of the inverse function, which may or may not be the desired value.

Let $f(z)$ be a single-valued function which is analytic except at a discrete set of points. For simplicity assume that f is analytic at $z=0$ with $f(0)=0$ and $f'(0)\neq 0$, so that $f(z)$ has a power-series representation around the origin of the form

$$f(z)=z\sum_{n=0}^{\infty}a_n z^n, \quad \text{for } |z|<\rho, \tag{1}$$

with $a_0\neq 0$. We assume that $\lim_{z\to\infty} f(z)=f_\infty$ ($z \in R^+$) exists and is finite. The problem is to find f_∞ from the coefficients a_n.

Bender and co-workers have proposed the following method. Write

$$f(z)=z\Big(\sum_{n=0}^{\infty} b_n z^n\Big)^{-1};$$
$$[f(z)]^N=z^N\Big(\sum_{n=0}^{\infty} c_n^{(N)} z^n\Big)^{-1}. \tag{2}$$

When an approximation is made by including only the first $(N+1)$ terms in the denominator of $[f(z)]^N$, the resulting rational function has a limit as $z\to\infty$; i.e.,

$$[f(z)]^N \approx z^N\Big(\sum_{n=0}^{N} c_n^{(N)} z^n\Big)^{-1} \to (c_N^{(N)})^{-1} \quad \text{as } z\to\infty. \tag{3}$$

Therefore define $Q_N=[c_N^{(N)}]^{-1/N}$ and define $f^*=\lim_{N\to\infty} Q_N$. The proposed solution is that $f_\infty=f^*$.

Our results are the following:

Proposition 1.—Let $w(f)=fz'(f)/z(f)$, where $z=f^{-1}$. Then

$$\liminf_{N\to\infty} |Q_N| = r, \tag{4}$$

where r is the radius of convergence of $w(f)$ around the point $f=0$ on the branch corresponding to $z(0)=0$.

Proof: Writing $c_N^{(N)}$ as a Cauchy integral and changing the variable of integration from z to f

we have

$$c_N^{(N)} = (2\pi i)^{-1} \oint_\gamma (z/f)^N z^{-(N+1)} dz = (2\pi i)^{-1} \int_{f(\gamma)} w f^{-(N+1)} df, \tag{5}$$

and so

$$w(f) = \sum_{N=0}^{\infty} c_N^{(N)} f^N. \tag{6}$$

Since $Q_N = (c_N^{(N)})^{-1/N}$, (4) follows from Hadamard's formula.

Proposition 2.—Every singular point (pole, essential singularity, or branch point) of $w(f)$ is a singular point of $z(f)$. Every nonzero singular point of $z(f)$ is a singular point of $w(f)$.

Proof: First note that $z(f)=0$ only if $f=0$ and that both z and w are analytic there. When $z(f) \neq 0$, $z(f)$ and $z'(f)/z(f)$ have the same singular points. So if f_0 is a singular point for $w(f) = fz'(f)/z(f)$ it is also one for $z'(f)/z(f)$ and $z(f)$. On the other hand, if $f_0 \neq 0$ is a singular point for $z(f)$, then it is also one for $z'(f)/z(f) = w(f)/f$ and $w(f)$.

Combining these two propositions we have the following:

Proposition 3.—Let r be the radius of convergence of $z(f)$ around the point $f=0$ on the branch corresponding to $z(0)=0$. Then

$$\liminf_{N\to\infty} |Q_N| = r. \tag{7}$$

Consequently the extrapolation procedure succeeds, in the sense that $\liminf_{N\to\infty} |Q_N| = |f_\infty|$, if and only if $r = |f_\infty|$.

Since f_∞ is a singular point of $z(f)$, the method will succeed if and only if it is the singularity of smallest modulus and is on the correct branch. This shows that the necessary condition found by Rivers[8,9] is also sufficient, if the condition that f_∞ be on the correct branch is included [River's saddle points are the singularities of $z(f)$]. For example, the method will work for $f(z) = z(z-a)(z-b)^{-1}(z-c)^{-1}$ if and only if $a(b+c-a) \geq 0$ (so that the singular point f_∞ is on the correct branch) and $1 \leq |1+(b+c)^{-2}\{[b(a-b)]^{1/2} \pm [c(a-c)]^{1/2}\}^2|$ (so that f_∞ is the smallest singular point). Also the method will never work if $f_\infty = 0$ [e.g., for $f(x) = xe^{-x}$] since the corresponding singularity cannot lie on the same sheet as the point $f=0$, $z=0$.

Equation (6), on which Proposition 1 is based, is similar to the Lagrange formula for reversion of a power series, which in our notation states that $z(f) = \sum_{N=1}^{\infty} N^{-1} C_{N-1}^{(N)} f^N$. This formula is easily proved by a slight modification of the derivation of Eq. (6). As before, a natural guess for $|f_\infty|$ is $\liminf_{N\to\infty} [C_{N-1}^{(N)}]^{1/N}$, the radius of convergence of $z(f)$ around $f=0$. This way of guessing f_∞ is similar to and yields the same limiting result as the method of Bender and co-workers.

This method might be improved by substituting for the line after (3) any of the several techniques in the literature for determining the smallest singular point of a series, e.g., ratio methods or Padé approximant methods, a review of which can be found in Hunter and Baker.[10] For many purposes these methods are better since they yield the precise location of the singular point rather than just its magnitude and often converge more quickly than the sequence $|Q_N|$.

Finally we remark that despite the limitations discussed above, this method has succeeded in a number of important problems (sometimes by approaching the correct value before diverging away and sometimes by the modified method). This success may be due to special properties of those problems. If so it would be important to understand these properties.

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