SIAM J. NUMER. ANAL.
Vol. 26, No. 5, pp. 1060-1080, October 1989

CONVERGENCE OF THE VORTEX METHOD FOR VORTEX SHEETS*

RUSSEL E. CAFLISCH†‡ AND JOHN S. LOWENGRUB†§

Abstract. Computation of the evolution of vortex sheets is delicate because of Kelvin-Helmholtz instability and singularity formation (infinite curvature). Convergence of the point vortex method and the vortex blob method is demonstrated for vortex sheets with both spatial and temporal discretization and with simulated roundoff error. The initial data is assumed to be a small analytic perturbation of a flat, uniform sheet. The proof works for a short time interval, certainly less than the first time of singularity formation. The analysis is performed in an analytic function space using the abstract Cauchy-Kowalewski Theorem. A numerical-analytic interpretation of analyticity is given.

Key words. vortex sheet, vortex method, numerical analysis, analyticity

AMS(MOS) subject classifications. 65M10, 76C05

1. Introduction. Computation of the evolution of vortex sheets in two-dimensional, incompressible inviscid flow is delicate because of Kelvin-Helmholtz instability and singularity formation (infinite curvature of the sheet). Numerical roundoff error can excite the physical instability to produce irregular results well before the physically correct singularity formation and roll-up of the sheet. Krasny [13], [14] has overcome this difficulty by two methods: a point vortex method with filtering to eliminate spurious high wavenumber components and a vortex blob method. A successful method based on series expansion was used by Meiron, Baker, and Orszag [15].

The aim of this paper is to prove convergence of the vortex method for a vortex sheet that is initially a small analytic perturbation of a flat, uniform sheet. The perturbation is required to be periodic for simplicity. The time interval for convergence is small. Certainly it does not include the first singularity time, so that the result does not include vortex sheet roll-up. Our convergence result applies to the point vortex method and to the vortex blob method, with spatial and temporal discretization and with simulated roundoff error. Stability for the point vortex method is maintained by the assumption of analyticity. The use of vortex blobs improves the stability, but decreases the accuracy of the method (see Theorem 2 for a precise statement).

Analyticity for the vortex sheet is needed to stabilize the Kelvin-Helmholtz instability. In fact the vortex sheet problem is known to be well posed in an analytic function space [5], [10], [17] but ill posed in certain nonanalytic spaces [6], [11]. The Cauchy-Kowalewski Theorem in abstract and discretized versions (see [2], [16], and Theorems 3 and 4 below) is the basic tool for construction of solutions in this paper.

One of the main contributions of this paper is to clarify the meaning of analyticity for numerical analysis: A function $f(z)$ is analytic in a strip $|\mathrm{Im}\, z|<\rho$ if (roughly speaking) its Fourier transform decays like $|\hat{f}(k)|< c\, e^{-\rho|k|}$. If the function is discretized with a length scale h, then the maximum wavenumber is $k_m = h^{-1}$. Analyticity of such discretized functions can be detected if the inequality on its transform can be verified.

* Received by the editors May 16, 1988; accepted for publication October 19, 1988. This research was supported in part by the Air Force Office of Scientific Research through University Research Initiative grant AFOSR-86-0352.

† Courant Institute of the Mathematical Sciences, New York University, 251 Mercer Street, New York, New York 10012.

‡ Present address, Mathematics Department, University of California at Los Angeles, Los Angeles, California 90024.

§ Present address, Mathematics Department, Stanford University, Stanford, California 94203.

This gives a restriction on the size of roundoff error e_r, that

$$|e_r| < c\, e^{-\rho/h} \tag{1.1}$$

for some constants c and ρ. This constraint on h and $|e_r|$ is our interpretation of analyticity for numerical analysis.

Constraint (1.1) is severe, but seems to be optimal in the presence of Kelvin-Helmholtz instability. It is in qualitative agreement with the numerical results of Krasny [13], [14], but no quantitative numerical study of the relation between e_r and h has been made. The constraint needs to be applied only to roundoff error, not to discretization error, since for analytic functions the discretization error is also analytic. For this problem the factor ρ will be proportional to UT, in which U is a characteristic velocity and T is the time over which convergence is proved. We expect this interpretation of analyticity to be useful for numerical analysis of many other ill-posed problems.

As a prerequisite to the convergence theorem, we prove global existence for analytic solutions to Krasny's desingularized vortex sheet equation (without discretization). This result is for arbitrary analytic initial data, without any assumed closeness to a flat sheet and does not use the Cauchy-Kowalewski Theorem. The same result was proved independently by Christoph Borgers [4]. Borgers also proved convergence for solutions of the discretized, desingularized equation, but with a rate that depends on the desingularization size δ, whereas our convergence result is uniform in δ.

The vortex blob method was first proposed in [7], and convergence of the blob method for smooth flow in $\mathbf{R}^2$ and $\mathbf{R}^3$ is proved in [1], [3], [8], [9], and [12]. Vortex sheet initial data is too singular to be included in those results, since they do not work in the presence of the Kelvin-Helmholtz instability. The results here are the first proof of convergence for the point vortex method in any context, and are made possible by the analytic function setting (for the sheet's profile).

The vortex sheet equations, some notation, and the main convergence result will be given in the next section. Global existence for the desingularized equations will also be stated in that section and proved in § 3. An outline of the convergence proof and a statement of the Cauchy-Kowalewski Theorem are presented in § 4. In § 5 several technical lemmas are proved. Then stability and consistency proofs are presented in §§ 6 and 7.

2. The vortex sheet equation and the convergence result. The position in the plane of a vortex sheet is described by the complex function $z(\gamma, t)$ in which $\gamma \in \mathbf{R}$ is the circulation variable. We will sometimes extend γ to be complex, as a mathematical construction. Evolution of the sheet is governed by the Birkhoff-Rott equation

$$\partial_t z^*(\gamma, t) = (2\pi i)^{-1} PV \int_{-\infty}^{\infty} (z(\gamma, t) - z(\gamma+\xi, t))^{-1}\, d\xi, \tag{2.1}$$

in which the integral is a principle value and $z^*(\gamma) = \overline{z(\bar{\gamma})}$. For γ real, $z^*(\gamma) = \overline{z(\gamma)}$.

An equilibrium solution for (2.1) is $z = \gamma$ corresponding to a flat sheet of uniform strength. For perturbations of a flat sheet, write $z = \gamma + s(\gamma, t)$, so that the Birkhoff-Rott equation becomes

$$\partial_t s^*(\gamma, t) = (2\pi i)^{-1} PV \int_{-\infty}^{\infty} (-\xi + s(\gamma, t) - s(\gamma+\xi, t))^{-1}\, d\xi. \tag{2.2}$$

If in addition s has period 2π in γ, then (2.2) becomes

$$\partial_t s^*(\gamma, t) = B[s](\gamma, t) \equiv (4\pi i)^{-1} PV \int_{-\pi}^{\pi} \cot \frac{1}{2}(-\xi + s(\gamma) - s(\gamma+\xi))\, d\xi. \tag{2.3}$$

Although the cotangent introduces some complexity, the bounded region of integration in (2.3) will simplify our analysis.

Next we define several approximations to (2.3) that result from desingularization by length scale δ, discretization in circulation variable γ by h, and discretization in time t by Δt. The solutions of the approximate equations are

s_δ: desingularized by δ,

s_D: desingularized by δ, discretized by h,

s_Δ: desingularized by δ, discretized by h and Δt.

They satisfy the equations

$$\partial_t s_\delta^*(\gamma, t) = B_\delta[s_\delta](\gamma, t), \tag{2.4}$$

$$\partial_t s_D^*(mh, t) = B_D[s_D](mh, t), \tag{2.5}$$

$$D_t s_\Delta^*(mh, (n+1)\Delta t) = B_D[s_\Delta](mh, n\Delta t) + e_1/\Delta t, \tag{2.6}$$

in which

$$B_\delta[s_\delta](\gamma, t) = (4\pi i)^{-1} \int_{-\pi}^{\pi} \frac{\cos \frac{1}{2}(s_\delta - s'_\delta - \xi) \sin \frac{1}{2}(s_\delta^* - s_\delta^{*\prime} - \xi)}{\sin \frac{1}{2}(s_\delta - s'_\delta - \xi) \sin \frac{1}{2}(s_\delta^* - s_\delta^{*\prime} - \xi) + \delta^2}\, d\xi, \tag{2.7}$$

$$B_D[s_D](mh, t) = (4\pi i)^{-1} h \sum_{j=1}^{N} \frac{\cos \frac{1}{2}(s_D - s'_D - jh) \sin \frac{1}{2}(s_D^* - s_D^{*\prime} - jh)}{\sin \frac{1}{2}(s_D - s'_D - jh) \sin \frac{1}{2}(s_D^* - s_D^{*\prime} - jh) + \delta^2}, \tag{2.8}$$

$$D_t s_\Delta^*(mh, n\Delta t) = \Delta t^{-1}(s_\Delta^*(mh, n\Delta t) - s_\Delta^*(mh, (n-1)\Delta t)). \tag{2.9}$$

In these equations N is a large integer, Δt and δ are small parameters, $h = 2\pi N^{-1}$, $1 \leqq m \leqq N$, and $n \geqq 0$. In (2.7), $s_\delta = s_\delta(\gamma, t)$, $s'_\delta = s_\delta(\gamma + \xi, t)$, and similarly for s_δ^*. In the sum in (2.8), $s_D = s_D(mh, t)$, $s'_D = s_D((m+j)h, t)$, and similarly for s_D^*, s_Δ^*. Initial data will be prescribed for (2.3), (2.4)-(2.6). The forcing term e_1 is included to simulate roundoff error. It is divided by Δt since D_t has been applied to the equation.

Note that (2.4) is a desingularization of (2.3) and is exactly (2.3) if $\delta = 0$. Equations (2.5) and (2.6) are vortex blob equations and are discretizations of (2.4). If $\delta = 0$, (2.5) and (2.6) become the equations for the point approximation to (2.3) (with and without time discretization), and the operator B_D becomes the point vortex operator

$$B_p[s_D](mh, t) = (4\pi i)^{-1} h \sum_{1}^{N} \cot \frac{1}{2}(s_D - s'_D - jh). \tag{2.10}$$

Existence and uniqueness for solutions of the periodic Birkhoff-Rott equation (2.3) are proved in [5], [17]. For the δ-equation (2.4) they are proved in Theorem 1 below. Existence and uniqueness for the discretized equations (2.5), (2.6) are easy to prove, since the vorticity is always positive and thus time invariance of the Hamiltonian $\sum \log(\sin(s(mh) - s((m+j)h) - jh))$ prevents singularities from forming.

Analysis of the discretized operator B_D uses the discrete Fourier Transform defined for discrete functions s_D by

$$\hat{s}_D^d(k) = N^{-1} \sum_{j=1}^{N} s_D(jh)\, e^{-ikjh}, \tag{2.11}$$

in which s_D is assumed to be 2π-periodic, $h = 2\pi/N$ and $k = 1, \cdots, N$, or equivalently, $k = -N/2+1, \cdots, N/2$ since $\hat{s}^d(k+N) = \hat{s}^d(k)$.

Convergence of the approximations above is stated using the following norms:

$$|||s_\delta|||_\rho = \sup_{|\mathrm{Im}\,\gamma| < \rho} |s_\delta(\gamma)|, \tag{2.12}$$

$$\|s_\delta\|_\rho = \sum_{-\infty}^{\infty} |\hat{s}_\delta(k)|\, e^{\rho|k|}, \tag{2.13}$$

$$\|s_D\|_{\delta\rho} = \sum_{-N/2+1}^{N/2} |\hat{s}^d_D(k)|\, e^{\rho\phi(k,\delta)}, \tag{2.14}$$

in which

$$\phi(k, \delta) = \min(|k|, \delta^{-2}). \tag{2.15}$$

Now we state the main results of this paper.

THEOREM 1 (Global existence for δ-equation). *Suppose that $s_0(\gamma) = s_\delta(\gamma, t=0)$ is analytic and 2π-periodic in γ and satisfies $|||s_0|||_{\rho_0} < R < \infty$, and that $0 < \delta < 1$. Then there is a unique periodic analytic solution $s_\delta(\gamma, t)$ for the desingularized vortex sheet equation* (2.4) *for all $t \geqq 0$. It satisfies*

$$|||s_\delta(t)|||_{\rho_1(t)} \leqq c\, e^{\lambda t/\delta^8} \tag{2.16}$$

in which $\rho_1(t) = c\delta^2 e^{-\lambda t/\delta^8}$, and c and λ are constants depending only on ρ_0 and R.

We note that Theorem 1 is not a near-linear result, as there is no smallness requirement on s_δ.

THEOREM 2 (Convergence of the vortex method). *Let κ be sufficiently small and let $s_0(\gamma) = s(\gamma, t=0)$ satisfy $\|s_0\|_{\rho_0} < \kappa < 1$ (in particular s_0 is analytic in γ). Let*

$$\begin{gathered} s_\delta(\gamma, t=0) = s_0(\gamma), \\ s_D(mh, t=0) = s_\Delta(mh, t=0) = s_0(mh) \end{gathered} \tag{2.17}$$

and assume that the roundoff error satisfies

$$|e_1| \leqq \varepsilon \Delta t \max(h, \delta^2)\, \max(e^{-\rho/h}, e^{-\rho/\delta^2}) \tag{2.18}$$

for all γ and t. Then for $\delta \geqq 0$, $h > 0$, and $\varepsilon \geqq 0$ sufficiently small (independent of the initial data), there are unique solutions s_Δ, s_D, and s_δ of (2.4)–(2.6) *with this initial data for $t < \alpha a \rho_0$. These solutions satisfy*:

(a) (*Convergence of δ-equation*)

$$\|s(t) - s_\delta(t)\|_{\rho(t)} \leqq c\delta; \tag{2.19}$$

(b) (*Convergence for spatial* (γ) *discretization*)

$$\|s(t) - s_D(t)\|_{\delta\rho(t)} \leqq c(\delta + h); \tag{2.20}$$

(c) (*Convergence for spatial and temporal discretization*)

$$\|s(t = n\Delta t) - s_\Delta(n\Delta t)\|_{\delta\rho(n\Delta t)} \leqq c(\delta + h + \varepsilon + \Delta t), \tag{2.21}$$

in which $\rho(t) = \alpha\rho_0 - a^{-1}t$. Then constant α can be any number less than 1, *and a depends only on α, κ, and ρ_0.*

The factors Δt and h are included in (2.18) because we have applied D_t and will apply $\mathcal{D}$ to the equation for s_Δ. They are insignificant compared to the exponential term. This theorem is uniform in δ and thus shows convergence of the point vortex and vortex blob methods for vortex sheets. The estimates (2.20) and (2.21) show that δ does not help accuracy. According to (2.18), inclusion of δ does allow larger roundoff error, since it helps stabilize the growth of the error. The point vortex method convergence is stated in (2.20), (2.21) with $\delta = 0$, which leads to the following corollary.

COROLLARY. *If s_p is the solution of the point vortex equation $D_t s_p = B_p[s_p] + e_1/\Delta t$, with roundoff error satisfying*

$$|e_1| \leqq \varepsilon \Delta t\, h\, e^{-\rho/h},$$

then

$$\|s(t) - s_p(t)\|_{\rho(t)} \leqq c(h + \varepsilon + \Delta t). \tag{2.22}$$

3. Global existence and uniqueness for the desingularized equation. Here we prove Theorem 1. The proof relies on an iteration scheme and does not use the Abstract Cauchy-Kowaleski Theorem. We show at each iteration that the approximate solution is analytic in $|\text{Im } \gamma| < \rho_1(t)$, as defined in the theorem, and satisfies the bound (2.16). The necessary introduction of trigonometric function, as in (2.3) for the periodic problem rather than (2.2), makes the proof complicated technically. This section is independent of the rest of the paper.

We introduce some space-saving notation and make some preparatory estimates. Denote

$$\begin{aligned}
E[s] &= |s| + |s'| + |s^*| + |s^{*\prime}|,\\
C[s] &= \sin\left(\frac{s - s' - \xi}{2}\right) \sin\left(\frac{s^* - s^{*\prime} - \xi}{2}\right) + \delta^2,\\
A_2[s] &= \left|\sin\left(\frac{s - s' - \xi}{2}\right)\right| + \left|\cos\left(\frac{s - s' - \xi}{2}\right)\right|,\\
A_1[s] &= A_2[s] + A_2[s^*]
\end{aligned}$$

in which $s = s(\gamma, t)$, $s' = s(\gamma + \xi, t)$, $s^* = s^*(\gamma, t)$, $s^{*\prime} = s^*(\gamma + \xi, t)$. We first estimate $A_1[s]$. We break this into two cases:

Case 1. If $|\sin((s - s' - \xi)/2)| < 1$, then clearly $A_2[s] < 2$.

Case 2. If $|\sin((s - s' - \xi)/2| > 1$ (i.e., if $s - s'$ has a large imaginary part), then $A_2[s] < 2|\sin((s - s' - \xi/2)| + 1$. Therefore, for all s,

$$A_1[s] < 4 + 2\left|\sin\left(\frac{s - s' - \xi}{2}\right)\right| + 2\left|\sin\left(\frac{s^* - s^{*\prime} - \xi}{2}\right)\right|. \tag{3.1}$$

Second, we estimate $C[s]^{-1}$ which is the denominator in the integral operator. On the real line $\text{Im } \gamma = 0$, $s^* = \bar{s}$, and $s^{*\prime} = \bar{s}'$ so that

$$C[s] = \left|\sin\left(\frac{s - s' - \xi}{2}\right)\right|^2 + \delta^2.$$

A similar estimate is needed off the real line, where $s^* \neq \bar{s}$. By a Taylor expansion in $\text{Im } \gamma$, it follows that for $|\text{Im } \gamma| \leqq \rho$

$$\begin{aligned}
|s^* - \bar{s}| &\leqq \rho \sup_{|\text{Im } \gamma| \leqq \rho} |s_\gamma|\\
&= \rho |||s_\gamma|||_\rho
\end{aligned}$$

so that

$$\begin{aligned}
\left|\sin\left(\frac{\bar{s} - \bar{s}' - \xi}{2}\right) - \sin\left(\frac{s^* - s^{*\prime} - \xi}{2}\right)\right| &\leqq \rho\left[\sup_{|\text{Im } \gamma| \leqq \rho} \left|\cos\left(\frac{s - s' - \xi}{2}\right)\right|\right] |||s_\gamma|||_\rho\\
&\leqq \rho\left(\sup_{|\text{Im } \gamma| \leqq \rho} A_1[s]\right) |||s_\gamma|||_\rho.
\end{aligned}$$

This estimate allows us to replace $\bar{s}$ and $\bar{s}'$ in the definition of $C[s]$ by s^* and $s^{*\prime}$ with a controllable error, so that $|C[s]|$ can be bounded below as

$$|C[s]| \geqq \frac{1}{2}\left|\sin\left(\frac{s^*-s^{*\prime}-\xi}{2}\right)\right|^2+\frac{1}{2}\left|\sin\left(\frac{s-s'-\xi}{2}\right)\right|^2-A_1^2\rho|||s_\gamma|||_\rho+\delta^2.$$

Combine this with (3.1) to obtain

$$(3.2)\quad |C[s]| \geqq \left\{\left|\sin\left(\frac{s^*-s^{*\prime}-\xi}{2}\right)\right|^2+\left|\sin\left(\frac{s-s'-\xi}{2}\right)\right|^2+\delta^2\right\}\left(\frac{1}{2}-\delta^{-2}48\rho|||s_\gamma|||_\rho\right).$$

Now we are ready to solve (5.4) by induction and the contraction mapping principle. Let

$$s_0(\gamma, t)=s_0(\gamma) \quad \forall t$$

and let

$$(3.3)\quad \begin{aligned} &\partial_t s_{n+1}^*(\gamma, t)=B_\delta[s_n](\gamma, t),\\ &s_{n+1}(\gamma, 0)=s_0(\gamma).\end{aligned}$$

Note that s_n is 2π-periodic in γ. We will show that

$$(3.4)\quad |||s_n(t)|||_{\rho_1(t)}+|||s_{n\gamma}(t)|||_{\rho_1(t)} \leqq c\, e^{\lambda t/\delta^8}$$

by induction, in which $\rho_1(t)$ is defined in Theorem 1. Suppose (3.4) is true for n and show it is true for $n+1$. Differentiate (3.3) or (2.4) in γ to obtain

$$(3.5)\quad \partial_t s_{n+1\gamma}^*(\gamma, t)=\frac{1}{4\pi i}\int_{-\pi}^{\pi} G_1[s_n]\, d\xi$$

where

$$\begin{aligned} G_1[s_n]=&\,k_1[s_n-s_n', s_n^*-s_n^{*\prime}](s_{n\gamma}-s_{n\gamma}')\\ &+k_2[s_n-s_n', s_n^*-s_n^{*\prime}](s_{n\gamma}^*-s_{n\gamma}^{*\prime}).\end{aligned}$$

Estimate

$$(3.6)\quad \begin{aligned} &k_1[s_n-s_n', s_n^*-s_n^{*\prime}]\\ &\quad=-\frac{1}{2}\frac{\sin^2((s_n^*-s_n^{*\prime}-\xi)/2)+\delta^2\sin((s_n-s_n'-\xi)/2)\sin((s_n^*-s_n^{*\prime}-\xi)/2)}{[\sin((s_n-s_n'-\xi)/2)\sin((s_n^*-s_n^{*\prime}-\xi)/2)+\delta^2]^2},\end{aligned}$$

$$(3.7)\quad k_2[s_n-s_n', s_n^*-s_n^{*\prime}]=\frac{\delta^2}{2}\frac{\cos((s_n-s_n'-\xi)/2)\cos((s_n^*-s_n^{*\prime}-\xi)/2)}{[\sin((s_n-s_n'-\xi)/2)\sin((s_n^*-s_n^{*\prime}-\xi)/2)+\delta^2]^2}.$$

It follows that

$$(3.8)\quad |k_1[s_n-s_n', s_n^*-s_n^{*\prime}]| \leqq \tfrac{1}{2}|C[s_n]|^{-2}A_1[s_n]^2,$$

$$(3.9)\quad |k_2[s_n-s_n', s_n^*]| \leqq \frac{\delta^2}{2}|C[s_n]|^{-2}A_1[s_n]^2$$

so that

$$(3.10)\quad |G_1[s_n]| \leqq |C[s_n]|^{-2}A_1[s_n]^2E[\partial_\gamma s_n].$$

Similarly,

$$(3.11)\quad \partial_t(s_{n+1}^*-s_n^*)_\gamma(\gamma, t)=\frac{1}{4\pi i}\int_{-\pi}^{\pi} G_2[s_n, s_{n-1}]\, d\xi$$

where

$$G_2[s_n, s_{n-1}] = G_1[s_n] - G_1[s_{n-1}],$$

i.e.,

$$\begin{aligned} G_2[s_n, s_{n-1}] = {} & k_1[s_n - s'_n, s^*_n - s'^*_n](s_{n\gamma} - s'_{n\gamma}) \\ & - k_1[s_{n-1} - s'_{n-1}, s^*_{n-1} - s^{*\prime}_{n-1}](s_{n-1\gamma} - s'_{n-1\gamma}) \\ & + k_2[s_n - s'_n, s^*_n - s^{*\prime}_n](s^*_{n\gamma} - s^{*\prime}_{n\gamma}) \\ & - k_2[s_{n-1} - s'_{n-1}, s^*_{n-1} - s^{*\prime}_{n-1}](s^*_{n-1\gamma} - s^{*\prime}_{n-1\gamma}). \end{aligned}$$

Therefore $G_2[s_n, s_{n-1}]$ is partially estimated by taking derivatives of $G_1[s_n]$, and hence of the products $k_1 \cdot (s_\gamma - s'_\gamma)$ and $k_2 \cdot (s^*_\gamma - s^{*\prime}_\gamma)$, with respect to $s_n - s'_n$ and $s^*_n - s^{*\prime}_n$. The result is that

$$\begin{aligned} |G_2[s_n, s_{n-1}]| \leqq {} & \sup_s \{C[s]^{-2} A_1[s]^2\} E[(s_n - s_{n-1})_\gamma] \\ & + \sup_s \{C[s]^{-4} A_1[s]^4\} (E[s_{n-1\gamma}] + E[s_{n\gamma}]) E[(s_n - s_{n-1})] \end{aligned} \tag{3.12}$$

in which the sup is taken over all $\Delta s = s - s'$ on the line segment connecting $\Delta s_1 = s_n - s'_n$ and $\Delta s_2 = s_{n-1} - s'_{n-1}$, and similarly for s^*.

Since by our induction hypothesis s_n and s_{n-1} (if $n \geqq 1$) satisfy (3.4), then for $|\mathrm{Im}\ \gamma| \leqq \rho_1(t)$, we have

$$|C[s_n](\gamma, t)| \geqq \frac{1}{2}\left[\left|\sin\left(\frac{s_n - s'_n - \xi}{2}\right)\right|^2 + \left|\sin\left(\frac{s^*_n - s^{*\prime}_n - \xi}{2}\right)\right|^2 + \delta^2\right] \tag{3.13}$$

and similarly

$$|C[s_{n-1}]| \geqq \frac{1}{2}\left[\left|\sin\left(\frac{s_{n-1} - s'_{n-1} - \xi}{2}\right)\right|^2 + \left|\sin\left(\frac{s^*_{n-1} - s^{*\prime}_{n-1} - \xi}{2}\right)\right|^2 + \delta^2\right]. \tag{3.14}$$

Using (3.13), (3.14), and (3.1) we obtain

$$|G_1[s_n]| \leqq c\delta^{-4} E[s_{n\gamma}], \tag{3.15}$$

$$|G_2[s_n]| \leqq c\delta^{-8}[(E[s_{n-1\gamma}] + E[s_{n\gamma}])E[s_n - s_{n-1}] + E[s_{n\gamma} - s_{n-1\gamma}]]. \tag{3.16}$$

Combine (3.15), (3.16) along with our induction hypothesis (3.4) and (3.5), (3.11) to get

$$|||s_{n+1\gamma}(t)|||_{\rho_1(t)} \leqq |||s_{0\gamma}|||_{\rho_0} + c\delta^{-4} \int_0^t |||s_{n\gamma}(t')|||_{\rho_1(t')}\, dt', \tag{3.17}$$

$$|||(s_{n+1} - s_n)_\gamma(t)|||_{\rho_1(t)} \leqq c\delta^{-8} \int_0^t [|||s_{n\gamma}|||_{\rho_1(t')} |||s_n - s_{n-1}|||_{\rho_1(t')} + |||s_{n\gamma} - s_{n-1\gamma}|||_{\rho_1(t')}]\, dt'. \tag{3.18}$$

It is clear that similar but simpler estimates hold for $|||s_{n+1}(t)|||_{\rho_1(t)}$ and $|||(s_{n+1} - s_n)(t)|||_{\rho_1(t)}$. Integration of (3.17) and (2.18) using (3.4) for n and large enough choice of λ, establishes (3.4) for $n+1$.

Now let

$$U_{n+1} = s_{n+1} - s_n, \tag{3.19}$$

$$V_{n+1} = s_{n+1\gamma} - s_{n\gamma} \tag{3.20}$$

so that for some λ

$$(3.21)\qquad |||U_{n+1}(t)|||_{\rho_1(t)} \leqq c\delta^{-2} \int_0^t |||U_n(t')|||_{\rho_1(t')}\, dt',$$

$$(3.22)\qquad |||V_{n+1}(t)|||_{\rho_1(t)} \leqq c\delta^{-8} \int_0^t [e^{\lambda t'/\delta^8} |||U_n(t')|||_{\rho_1(t')} + |||V_n(t')|||_{\rho_1(t')}]\, dt'.$$

Define the following norm:

$$(3.23)\qquad \|U, V\| = \sup e^{-\lambda_1 t} |||U|||_{\rho_1(t)} + \sup_t e^{-\lambda_2 t} |||V|||_{\rho_1(t)}.$$

From (3.21), (3.22) it follows that

$$\|U_{n+1}, V_{n+1}\|_1 \leqq c[\{\delta^{-2}/\lambda_1 + \delta^{-8}(\lambda\delta^{-8}+\lambda_1)^{-1} \exp((\lambda\delta^{-8}+\lambda_1-\lambda_2)t)\} \cdot \|U_n\| + (\delta^{-8}/\lambda_2)\|V_n\|_1].$$

Consequently, by proper choice of λ_1 and λ_2, we obtain

$$(3.24)\qquad \|U_{n+1}, V_{n+1}\| \leqq \tfrac{1}{2}\|U_n, V_n\|_1.$$

This shows that the sequence s_n has a unique limit s. Since s is the uniform limit of analytic functions s_n, it is analytic in $|\mathrm{Im}\ \gamma| \leqq \rho_1(t)$.

4. Outline of convergence proof. The proof of Theorem 2 involves the usual estimates of consistency and stability errors as well as roundoff error. The effect of these errors is stabilized by the assumption of analyticity. The basic tool for analyzing the resulting solutions is the Cauchy-Kowalewski Theorem. Since the operators in the Birkhoff-Rott equation are not differential operators, we use the Abstract Cauchy-Kowalewski Theorem [2], [16], as well as a new but simple generalization to discrete time.

THEOREM 3 (Cauchy-Kowalewski Theorem, continuous time). *Let $\mathcal{B}_\rho$, with norm $\|\cdot\|_\rho$ for $\rho > 0$, be a scale of Banach spaces satisfying $\mathcal{B}_{\rho'} \subset \mathcal{B}_\rho$, $\|\cdot\|_{\rho'} > \|\cdot\|_\rho$ for $\rho' > \rho > 0$. Suppose that $A(u, t)$ satisfies*

(i) *There are constants C, R, K, ρ_0 such that for any $0 < \rho < \rho' < \rho_0$, $A(u, t)$ is a continuous mapping of $\{u \in \mathcal{B}_{\rho'}: \|u\|_{\rho'} < R\} \times \{t\}$ into $\mathcal{B}_\rho$.*

(ii) *For any $\rho < \rho' < \rho_0$, any $u, v \in \mathcal{B}_\rho$ with $\|u\|_\rho < R$, $\|v\|_\rho < R$, and any t,*

$$(4.1)\qquad \|A(u, t) - A(v, t)\|_\rho \leqq C(\rho' - \rho)^{-1}\|u - v\|_{\rho'}.$$

(iii) *$A(0, t)$ is continuous in t with values in $\mathcal{B}_\rho$ for every $0 < \rho < \rho_0$ and satisfies*

$$(4.2)\qquad \|A(0, t)\|_\rho \leqq K(\rho_0 - \rho)^{-1}.$$

Then the equation

$$(4.3)\qquad \frac{d}{dt} u(t) = A(u, t), \qquad u(0) = 0$$

has a unique solution $u(t) \in \mathcal{B}_{\rho(t)}$, with $\|u(t)\|_{\rho(t)} < R$ for $|t| < a\rho_0$ and $\rho(t) = \rho_0 - a^{-1}|t|$, in which the inverse speed is $a = c \min(C^{-1}, R/K)$ and c is an absolute constant.

THEOREM 4 (Cauchy-Kowalewski Theorem, discrete time). *Let* $0<\Delta t<1$. *Under the same hypotheses as Theorem* 3, *the discrete-time equation* (*with* D_t *defined in* (2.9))

$$D_t u(n\Delta t)=A(u,n\Delta t),\qquad u(0)=0 \tag{4.4}$$

has a solution $u(n\Delta t)\mathscr{B}_{\rho(t)}$ *with* $\|u(n\Delta t)\|_{\rho(t)}<R$ *for* $|n|\Delta t<a\rho_0$ *and* $\rho(t)=\rho_0-a^{-1}|n|\Delta t$.

Note that the results of Theorem 4 are independent of the discretization size Δt.

Estimates for (2.3)-(2.6) will be found after differentiating the equations. The usefulness of this differentiation is like that of differentiating a nonlinear differential equation to get a quasilinear equation. For the spatially discrete (in γ) equations (2.5), (2.6) it is convenient to introduce the finite difference operator $\mathscr{D}$ based on discrete Fourier transform by

$$\mathscr{D}s=(i\operatorname{sgn}(k)\phi(k,\delta)\hat{s}^d(k))^{\check{}d}, \tag{4.5}$$

in which $\phi(k,\delta)=\min(|k|,\delta^{-2})$ as in (2.15). This represents a cutoff derivative.

Now we begin the proof of Theorem 2. By recombining (2.3)-(2.6) and applying ∂_γ or $\mathscr{D}$ to them, we obtain the following equations for s, s_δ, s_D, and s_Δ:

$$\partial_t s_\gamma=\partial_\gamma B[s], \tag{4.6}$$

$$\partial_t(s-s_\delta)_\gamma=\partial_\gamma(B[s]-B_\delta[s])+\partial_\gamma(B_\delta[s]-B_\delta[s_\delta]), \tag{4.7}$$

$$\partial_t\mathscr{D}(s-s_D)=\mathscr{D}(B[s]-B_D[s])+\mathscr{D}(B_D[s]-B_D[s_D]), \tag{4.8}$$

$$D_t\mathscr{D}(s-s_\Delta)=(D_t-\partial_t)\mathscr{D}s+\mathscr{D}(B[s]-B_D[s])+\mathscr{D}(B_D[s]-B_D[s_\Delta])-\mathscr{D}e_1/\Delta t \tag{4.9}$$

with initial data

$$s_\gamma(\gamma,0)=s_{0\gamma}(\gamma), \tag{4.10}$$

$$(s-s_\delta)_\gamma(\gamma,0)=0, \tag{4.11}$$

$$\mathscr{D}(s-s_D)=0, \tag{4.12}$$

$$\mathscr{D}(s-s_\Delta)=0. \tag{4.13}$$

By the first assumption of Theorem 2, $\|s_{0\gamma}\|_{\rho_0}<\kappa<1$. Thus the analytic existence results of [5], [14] show that the Birkhoff-Rott equation (4.6), (4.10) has a solution $s(\gamma,t)$ with $\|s_\gamma(t)\|_{\rho_2(t)}<\kappa$ for $|t|<a_2\rho_0$ and $\rho_2(t)=\rho_0-t/a_2$. The other assumptions of Theorem 2 imply that

$$\|\mathscr{D}e_1/\Delta t\|_{\delta\rho_0}<\varepsilon. \tag{4.14}$$

In the next three sections, we prove the following bounds.

Consistency bounds. For any $\rho'>\rho>0$, any $\delta\geqq 0$, any $h=2\pi/N$, and any s satisfying $\|\partial_\gamma s\|_{\rho'}<\kappa$,

$$\|\partial_\gamma(B[s]-B_\delta[s])\|_\rho\leqq c\delta(\rho'-\rho)^{-3}\|\partial_\gamma s\|_{\rho'}, \tag{4.15}$$

$$\|\mathscr{D}(B[s]-B_D[s])\|_{\delta\rho}\leqq c(h+\delta)(\rho'-\rho)^{-3}\|\partial_\gamma s\|_{\rho'}. \tag{4.16}$$

If s satisfies (2.3) then

$$\|(\partial_t-D_t)\mathscr{D}s\|_{\delta\rho}\leqq c\Delta t(\rho'-\rho)^{-2}\|\partial_\gamma s\|_{\rho'}. \tag{4.17}$$

In particular if s is an exact solution of the vortex sheet equation (2.3) with $\|\partial_\gamma s\|_{\rho_2(t)} < \kappa < 1$, then for $\alpha\rho_2(t) > \rho > 0$ (so that we may take $\rho' = \rho_2(t)$ in (4.15)–(4.17)),

$$\|\partial_\gamma(B[s] - B_\delta[s])\|_{\delta\rho} \leqq c\delta(\rho_0 - \rho)^{-1}, \tag{4.15'}$$

$$\|\mathscr{D}(B[s] - B_D[s])\|_{\delta\rho} \leqq c(h+\delta)(\rho_0 - \rho)^{-1}, \tag{4.16'}$$

$$\|(\partial_t - D_t)\mathscr{D}s\|_{\delta\rho} \leqq c\Delta t(\rho_0 - \rho)^{-1} \tag{4.17'}$$

for any number $\alpha < 1$, with c a constant depending on α.

Stability bounds. For any $\rho' > \rho > 0$, any $\delta \geqq 0$, and any s, $\tilde{s}$ satisfying $\|\partial_\gamma s\|_{\rho'} < \kappa$, $\|\partial_\gamma \tilde{s}\|_{\rho'} < \kappa$,

$$\|\partial_\gamma(B_\delta[s] - B_\delta[\tilde{s}])\|_\rho \leqq c(\rho' - \rho)^{-1}\|\partial_\gamma(s - \tilde{s})\|_{\rho'}. \tag{4.18}$$

For any such ρ, ρ', and δ and any s, $\tilde{s}$ satisfying $\|\mathscr{D}s\|_{\delta\rho'} < \kappa$, $\|\mathscr{D}\tilde{s}\|_{\delta\rho'} < \kappa$,

$$\|\mathscr{D}(B_D[s] - B_D[\tilde{s}])\|_{\delta\rho} \leqq c(\rho' - \rho)^{-1}\|\mathscr{D}(s - \tilde{s})\|_{\delta\rho'}. \tag{4.19}$$

Discretization bounds. For any $\rho' > \rho > 0$, any $\delta \geqq 0$ and any s,

$$\|\mathscr{D}s\|_{\delta\rho} \leqq \|\partial_\gamma s\|_\rho, \tag{4.20}$$

$$\|s\|_{\delta\rho} \leqq \|s\|_\rho. \tag{4.21}$$

Now we are ready to apply the Cauchy-Kowalewski Theorems 3 and 4 to solving the equations above.

(a) *δ-equation.* To solve (4.7) (or (2.4)) for s_δ with s known, set

$$u = \partial_\gamma(s - s_\delta), \tag{4.22}$$

$$A(u, t) = \partial_\gamma(B[s] - B_\delta[s]) + \partial_\gamma(B_\delta[s] - B_\delta[s - \textstyle\int u]). \tag{4.23}$$

Assumptions (i)–(iii) of Theorem 3, with $R = K = c\delta$, $C = c$, are implied by estimates (4.15′), (4.18), which yield

$$\begin{aligned} &\|A(0, t)\|_\rho < c\delta(\rho_0 - \rho)^{-1}, \\ &\|A(u) - A(\bar{u})\|_\rho < c(\rho' - \rho)^{-}\|u - \bar{u}\|_{\rho'} \end{aligned} \tag{4.24}$$

for $\alpha\rho_2(t) > \rho' > \rho > 0$, with $\alpha < 1$. Therefore a solution $u = (s - s_\delta)_\gamma$ for (4.7), (4.11) exists for $t \leqq \alpha a\rho_0$ with $\rho(t) = \alpha\rho_0 - a^{-1}t$ and $\|u\|_{\rho(t)} \leqq c\delta$, i.e., with

$$\|(s - s_\delta)(t)\|_{\rho(t)} \leqq c\delta. \tag{4.25}$$

(b) *Space discretization.* To solve (4.8) for $\mathscr{D}s$, set

$$\begin{aligned} &u = \mathscr{D}(s - s_D), \\ &A(u, t) = \mathscr{D}(B[s] - B_D[s]) + \mathscr{D}(B_D[s] - B_D[s - \mathscr{D}^{-1}u]). \end{aligned} \tag{4.26}$$

$\mathscr{D}^{-1}$ is well defined since k is discrete and $\hat{s}(0)$ can be set to 0. Use the norm $\|\cdot\|_{\delta\rho}$ and bound $\|\mathscr{D}s\|_{\delta\rho} \leqq \|\partial_\gamma s\|_\rho$ by (4.20). Assumptions (i)–(iii) are implied, with $R = K = c(h+\delta)$, $C = c$, by the estimates (4.16′) and (4.19), i.e.,

$$\begin{aligned} &\|A(0, t)\|_{\delta\rho} \leqq c(h+\delta)(\rho_0 - \rho)^{-1}, \\ &\|A(u, t) - A(\tilde{u}, t)\|_{\delta\rho} \leqq c(\rho' - \rho)^{-1}\|u - \tilde{u}\|_{\delta\rho'} \end{aligned} \tag{4.27}$$

for $\alpha\rho_2(t) > \rho' > \rho > 0$. Therefore a solution $u = \mathscr{D}(s - s_D)$ for (4.8), (4.12) exists for $t \leqq \alpha a \rho_0$ and $\|u\|_{\delta\rho(t)} \leqq c(h+\delta)$, i.e., with

$$\|(s - s_D)(t)\|_{\delta\rho(t)} \leqq c(h+\delta). \tag{4.28}$$

(c) *Space and time discretizations.* To solve (4.9) for s_Δ, set

$$\begin{aligned} &u = \mathscr{D}(s - s_\Delta),\\ &A(u, t) = (\partial_t - D_t)\mathscr{D}s + \mathscr{D}(B[s] - B_D[s]) + \mathscr{D}(B_D[s] - B_D[s - \mathscr{D}^{-1}u]) + \mathscr{D}e_1/\Delta t. \end{aligned} \tag{4.29}$$

Use the norm $\|\cdot\|_{\delta\rho}$. Estimates (4.14), (4.17′), (4.19), and (4.16′) imply that

$$\begin{aligned} &\|A(0, t)\|_{\delta\rho} \leqq c(h+\delta+\Delta t+\varepsilon)(\rho_0 - \rho)^{-1},\\ &\|A(u, t) - A(\tilde{u}, t)\|_{\delta\rho} < c(\rho' - \rho)^{-1}\|u - \tilde{u}\|_{\delta\rho'}. \end{aligned} \tag{4.30}$$

By Theorem 4, with $R = K = c(h+\delta+\Delta t+\varepsilon)$, $C = c$, there is a solution $u = \mathscr{D}(s - s_D)$ for (4.9), (4.13) for $t \leqq \alpha a \rho_0$ and $\|u\|_{\delta\rho(t)} \leqq c(h+\delta+\Delta t+\varepsilon)$, i.e., with

$$\|s - s_\Delta\|_{\delta\rho(t)} \leqq c(h+\delta+\Delta t+\varepsilon). \tag{4.31}$$

Combination of results (4.25), (4.28), and (4.31) yields the results (2.19)-(2.21) of Theorem 2. Therefore the proof of Theorem 2 is complete once the inequalities (4.15)-(4.21) are verified.

5. Technical lemmas. In this section several technical lemmas are presented on the discrete Fourier transform, the analytic norms, a trigonometric integral, an expansion of the integrand or summand of B_δ or B_D, and the trapezoidal rule. The first two lemmas present basic facts about the discrete Fourier transform and the analytic norms.

LEMMA 1. (Discrete Fourier Transform). *Let $\|s_\gamma\|_\rho < \infty$. Then for $-N/2 \leqq k \leqq N/2$,*

$$|(\hat{s} - \hat{s}^d)(k)| \leqq e^{-\rho N/2} \min(\|s\|_\rho, N^{-1}2\|s_\gamma\|_\rho). \tag{5.1}$$

Proof. Let $h = 2\pi/N$. Estimate

$$\begin{aligned} |(\hat{s} - \hat{s}^d)(k)| &= |\hat{s}(k) - N^{-1}\sum_{j=1}^{N} e^{-ikjh}s(jh)|\\ &= \left|\hat{s}(k) - N^{-1}\sum_{l=-\infty}^{\infty}\hat{s}(l)\left(\sum_{j=1}^{N} e^{-2\pi i(k-l)j/N}\right)\right|\\ &\leqq \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} |\hat{s}(k+mN)|\\ &\leqq e^{-\rho N/2}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} |\hat{s}(k+mN)|\, e^{\rho|k+mN|}\\ &\leqq e^{-\rho N/2}\min\{\|s\|_\rho, 2N^{-1}\|s_\gamma\|_\rho\}, \end{aligned} \tag{5.2}$$

which finishes the proof of Lemma 1. □

LEMMA 2. (The norm $\|\cdot\|_{\delta\rho}$.) *For any $\delta \geqq 0$ and $\rho' > \rho > 0$,*

$$\|fg\|_{\delta\rho} \leqq \|f\|_{\delta\rho}\|g\|_{\delta\rho}, \tag{5.3}$$

$$\|\partial_\gamma f\|_\rho \leqq c_1(\rho' - \rho)^{-1}\|f\|_{\rho'}, \tag{5.4}$$

$$\|\mathscr{D}f\|_{\delta\rho} \leqq c_2(\rho' - \rho)^{-1}\|f\|_{\delta\rho'}, \tag{5.5}$$

$$\|\mathscr{D}f\|_{\delta\rho} \leqq \|f_\gamma\|_\rho. \tag{5.6}$$

Proof. (i) Since $\phi(k,\delta)=\min(|k|,\delta^{-2})$, then

$$\phi(m+n,\delta)\leqq\phi(m,\delta)+\phi(n,\delta). \tag{5.7}$$

Since the convolution formula works for the discrete Fourier transform, it follows that

$$\begin{aligned}\|fg\|_{\delta\rho}&=\sum_k\Big|\sum_l\hat f^d(l)\hat g^d(k-l)\,e^{\rho\phi(k,\delta)}\Big|\\&\leqq\sum_l\sum_m|\hat f^d(l)|\,|\hat g^d(m)|\,e^{\rho\phi(l,\delta)}\,e^{\rho\phi(m,\delta)}\\&\leqq\|f\|_{\delta\rho}\|g\|_{\delta\rho}.\end{aligned}$$

(ii) First bound $\phi(k,\delta)\,e^{-\Delta\rho\phi(k,\delta)}\leqq c\Delta\rho^{-1}$ with $\Delta\rho=\rho'-\rho$. Thus

$$\begin{aligned}\|\mathscr{D}f\|_{\delta\rho}&\leqq\sum_{|k|\leqq N/2}\phi(k,\delta)|\hat f^d(k)|\,e^{\rho\phi(k,\delta)}\\&\leqq\sup_{|k|\leqq N/2}\big(\phi(k,\delta)\,e^{-\Delta\rho\phi(k,\delta)}\big)\sum_{|k|\leqq N/2}|\hat f^d(k)|\,e^{\rho'\phi(k,\delta)}\\&\leqq c(\rho'-\rho)^{-1}\|f\|_{\delta\rho'}.\end{aligned}$$

Estimate (5.4) is derived in a similar way.

(iii) Use an intermediate step of (5.2) to estimate

$$\begin{aligned}\|\mathscr{D}f\|_{\delta\rho}&\leqq\sum_{|k|\leqq N/2}\phi(k,\delta)|\hat f^d(k)|\,e^{\rho\phi(k,\delta)}\\&\leqq\sum_{|k|\leqq N/2}|k|(|\hat f(k)|+|(\hat f^d-\hat f)(k)|)\,e^{\rho|k|}\\&\leqq\sum_{|k|\leqq N/2}|k|\,e^{\rho|k|}\Big(|\hat f(k)|+\sum_{\substack{m=-\infty\\m\neq0}}^{\infty}|\hat f(k+mN)|\Big)\\&\leqq\sum_{k=-\infty}^{\infty}|k|\,e^{\rho|k|}|\hat f(k)|\\&=\|f_\gamma\|_\rho,\end{aligned}$$

which finishes the proof of Lemma 2. □

Estimation of the operator B, its approximations, and their differences is performed in the Fourier norms. In order to deal with the nonlinearity of the kernel of B, B_δ, and B_D with the Fourier transform, we expand the integrand in power series in $(s-s')$ and $(s^*-s^{*\prime})$. In other words, we wish to express B_δ, B and B_D as a sum of integrals of the form

$$F_\delta^{m,n,r}[s](\gamma,t)=\int_{-\pi}^{\pi}(s-s')^m(s^*-s^{*\prime})^n\frac{(\sin\xi/2)^r\psi(\xi)}{(\sin^2\xi/2+\delta^2)^N}\,d\xi, \tag{5.8}$$

where $m+n+r=2N-1$ is odd and $\psi(\xi)$ is a smooth function in ξ and is also even. This allows for the easy calculation of the Fourier transform:

$$\begin{aligned}\hat F_\delta^{m,n,r}[s](k,t)&=\sum_{l_1+\cdots+l_{m+n}=k}\prod_{j=1}^{m}\hat s(l_j)\prod_{j=m+1}^{m+n}\hat s^*(l_j)\\&\quad\cdot\int_{-\pi}^{\pi}\prod_{j=1}^{m+n}(1-e^{il_j\xi})\frac{(\sin\xi/2)^r\psi(\xi)}{(\sin^2\xi/2+\delta^2)^N}\,d\xi\\&=\sum_{l_1+\cdots+l_{m+p}=k}\prod_{j=1}^{m}\hat s(l_j)\prod_{j=m+1}^{m+n}\hat s^*(l_j)\\&\quad\cdot(i2)^{m+n+1}\int_0^{\pi}\prod_{j=1}^{2N}\Big(\sin\frac{l_j\xi}{2}\Big)\frac{\psi(\xi)}{(\sin^2\xi/2+\delta^2)^N}\,d\xi,\end{aligned} \tag{5.9}$$

in which $l_{m+n+1}=\cdots=l_{2N-1}=1$ and $l_{2N}=k$. Thus we are led to an estimate of a trigonometric integral in order to estimate $\|F_\delta^{m,n,p}[s]\|_{\delta\rho}$.

We will first give an expansion lemma, which will tell us how to expand the kernels of B, B_δ, B_D. Then we will give the trigonometric integral lemma.

LEMMA 3. (Expansion lemma for B, B_δ, B_D, B_p.) *Let x, y, w, ξ satisfy $|x|<|\xi|/2$, $|y|<|\xi|/2$, $|\xi|<\pi$ and $|w|<1$. Define*

$$F(x, y, \xi, \delta)=\frac{\cos(x-\xi)/2\sin(y-\xi)/2}{\sin(x-\xi)/2\sin(y-\xi)/2+\delta^2}.$$

Then

$$F(x,y,\xi,\delta)-F(x,y,\xi,0)=\sum_{N=1}^{\infty}\sum_{m+n+r=2N-1}^{\infty}x^m y^n\left(\sin\frac{\xi}{2}\right)^r \tag{5.10}$$
$$\cdot\,[a_{mnr}(\xi,\delta)(\sin^2\xi/2+\delta^2)^{-N}-a_{mnr}(\xi,0)\,(\sin\xi/2)^{-2N}],$$

$$F(x,y,\xi,\delta)-F(\tilde{x},\tilde{y},\xi,\delta)=\sum_{N=1}^{\infty}\sum_{m+n+p+q+r=2N-2}^{\infty}[a_{mnpqr}(\xi)(x-\tilde{x})+b_{mnpqr}(\xi)(y-\tilde{y})] \tag{5.11}$$
$$\cdot\,x^m y^n\tilde{x}^p\tilde{y}^q(\sin\xi/2)^r(\sin^2\xi/2+\delta^2)^{-N},$$

$$\cot\frac{1}{2}(-x-\xi)+(1+w)^{-1}\cot\xi/2 \tag{5.12}$$
$$=\sum_{m,n=0}^{\infty}a_{mn}\frac{x+2w\sin\xi/2}{\sin^{m+2}\xi/2}x^m w^n+\sum_{m=0}^{\infty}b_m\frac{x^m}{(\sin\xi/2)^{m-1}}$$

in which a_{mnr}, a_{mnpqr}, b_{mnpqr}, a_{mn}, b_m are smooth functions of ξ and δ, which are even in ξ with

$$\sum_{m+n+p+q=M}\sum_{r=1}^{\infty}c^r[|a_{mnpqr}|+|a'_{mnpqr}|+|b_{mnpqr}|+|b'_{mnpqr}|]<\tilde{c}^M,$$

$$\sum_{m+n=M}\sum_{r=1}^{\infty}c^r[|a_{mnr}|+|a'_{mnr}|]<\tilde{c}^M,$$

$$a_{mn}(\xi,\delta)-a_{mn}(\xi,0)=\delta^2 d(\xi,\delta)$$

for any c where d is smooth in ξ and δ and $\tilde{c}$ depends only on c.

The proof of this lemma is quite tedious and uninstructive. It involves products of expansions, repeated use of the binomial theorem, and various trigonometric identities. The main idea, however, is quite general and can be used in the context of many other problems. The idea simply is that since the kernels are analytic in the x, y, ω variables, we can express them as power series in these variables. In our convergence proof, we will have

$$x=s-s',\quad y=s^*-s^{*\prime},\quad w=s_\gamma. \tag{5.13}$$

We emphasize that this lemma is merely a tool for obtaining bounds on B, B_δ, B_D, and our argument depends only on the existence of such an expansion and not on its exact form.

As we have seen in (5.9), the estimation of the Fourier transform of B, B_δ, B_D requires the estimate of several trigonometric integrals. We therefore need the following lemma.

LEMMA 4. (Trigonometric integrals). *Let $\delta>0$, $k_1,\cdots,k_{2n}$, k be any numbers and ψ a smooth function which is even in ξ. Then,*

$$\left|\int_{-\pi}^{\pi}\prod_{j=1}^{n}\left(\frac{e^{ik_j\xi}-1}{\sin\xi/2}\right)\psi(\xi)\,d\xi\right|\leqq c\prod_{j=1}^{n}|k_j|, \tag{5.14}$$

$$\left|\int_{-\pi}^{\pi}\left(\prod_{m=1}^{N}\frac{e^{ik_m\xi}-1}{\sin\xi/2}\right)\left(\frac{e^{ik\xi}-1-2ik\sin\xi/2}{\sin^2\xi/2}\right)\psi(\xi)\,d\xi\right|<c|k|^2\prod_{m=1}^{N}|k_m|, \tag{5.15}$$

$$\left|\int_{0}^{\pi}\left(\prod_{m=1}^{2N}\sin k_m\xi\right)\{(\sin^2\xi/2+\delta^2)^{-N}-(\sin\xi/2)^{-2N}\}\psi(\xi)\,d\xi\right|<cN\delta\prod_{m=1}^{2N}|k_m|, \tag{5.16}$$

$$\left|\int_{0}^{\pi}\left(\prod_{m=1}^{N}\frac{\sin k_m\xi/2}{(\sin^2\xi/2+\delta^2)^{1/2}}\right)\psi(\xi)\,d\xi\right|\leqq c\prod_{m=1}^{N-1}\phi(k_m,\delta), \tag{5.17}$$

$$\left|\sum_{j=0}^{N}\left(\prod_{m=1}^{N}\frac{\sin(k_mjh/2)}{(\sin^2(jh/2)+\delta^2)^{1/2}}\right)\psi(jh)\right|\leqq c\prod_{m=1}^{n-1}\phi(k_m,\delta), \tag{5.18}$$

in which $\phi(k,\delta)=\min(|k|,\delta^{-2})$ as before and $c<$ constant $\sup_\xi(|\psi|+|\psi'|)$.

Remarks. The first three estimates (5.14)-(5.16) of Lemma 4 are used for consistency and are not delicate. The last two estimates (5.17) and (5.18) are used for stability bounds and require more care. As will be seen in the next section, these last two estimates show that B is no worse than a derivative. We also believe that this result could be improved by replacing δ^{-2} by δ^{-1} in the definition of ϕ, but we were unable to prove this.

Proof of Lemma 4. The first three estimates are straightforward. Denote the integral on the left side of (5.17) as $I_n(\psi,k,\cdots,k_n)$. We will obtain estimate (5.18) by induction. We first estimate I_1. Replace ψ by the more general function

$$\tilde\psi=\psi\xi^r(\sin^2\xi/2+\delta^2)^{-r/2}. \tag{5.19}$$

To estimate I_1, denote $[x]=$ integer part of x, and

$$f(\xi)=\frac{\tilde\psi(2\xi)}{(\sin^2\xi+\delta^2)^{1/2}}.$$

Decompose I_1 as

$$I_1(\tilde\psi)=J_1+J_2+J_3, \tag{5.20}$$

in which

$$\begin{aligned}|J_1|&=\left|\int_0^{4\pi/k_1}\frac{\sin(k_1\xi/2)}{(\sin^2\xi/2+\delta^2)^{1/2}}\tilde\psi(\xi)\,d\xi\right|\\&\leqq\int_0^{4\pi/k_1}\frac{k_1\xi}{\xi}|\psi(\xi)|\,d\xi\\&\leqq 2\pi\sup|\psi|,\end{aligned} \tag{5.21}$$

$$\begin{aligned}|J_2|&=\left|\int_{[k_1/4]4\pi/k_1}^{\pi}\frac{\sin k_1\xi/2}{(\sin^2\xi/2+\delta^2)^{1/2}}\tilde\psi(\xi)\,d\xi\right|\\&\leqq\left|2\int_{[k_1/4]2\pi/k_1}^{\pi/2}\frac{\sin k_1\xi}{(\sin^2\xi/2+\delta^2)^{1/2}}\psi(2\xi)\,d\xi\right|\\&\leqq 2\int_{[k_1/4]2\pi/k_1}^{\pi/2}\frac{1}{(\frac14+\delta^2)^{1/2}}|\psi|\,d\xi<c\sup|\psi|\end{aligned} \tag{5.22}$$

since $|\pi/2-[k_1/4](2\pi/k_1)|<\pi/4$, and

$$\begin{aligned}
|J_3| &= \left|\int_{4\pi/k_1}^{[k_1/4](4\pi/k_1)} \sin(k_1\xi/2) f(\xi/2)\, d\xi\right| \\
&= \left|2\sum_{j=1}^{[k_1/4]} \int_{2\pi j/k_1}^{2\pi(j+1)/k_1} \sin(k_1\xi) f(\xi)\, d\xi\right| \\
&\leqq 2\sum_{j=1}^{[k_1/4]} \int_{2\pi j/k_1}^{\pi(2j+1)/k_1} |\sin(k_1\xi)|\, |f(\xi)-f(\xi+\pi/k_1)|\, d\xi \\
&\leqq 2\sum_{j=1}^{[k_1/4]} \int_{2\pi j/k_1}^{\pi(2j+1)/k_1} \pi k_1^{-1} |\sup f'(\xi)|\, d\xi \\
&\leqq 2\sum_{j=1}^{[k_1/4]} \left(\frac{\pi}{k_1}\right)^2 \sup_{2\pi j/k_1<\xi<\pi(2j+1)/k_1} |f'(\xi)| \qquad (5.23)\\
&\leqq ck_1^{-2} \sup(|\psi|+|\psi'|) \sum_{j=1}^{[k_1/4]} \sup\left(\frac{(2\pi j/k_1)^{r+1}}{(\sin^2(2\pi j/k_1)^{r+1}+\delta^2)^{(r+3)/2}}\right) \\
&\leqq ck_1^{-1} \sup(|\psi|+|\psi'|) \int_{1/k_1}^{1/4} \frac{x^{(r+1)}}{(x^2+\delta^2)^{(r+3)/2}}\, dx \\
&\leqq c \sup(|\psi|+|\psi'|)(k_1\delta^{-1}) \int_{1/k_1\delta}^{1/4\delta} \frac{y}{(y^2+1)}\, dy \\
&\leqq c \sup(|\psi|+|\psi'|).
\end{aligned}$$

Combine (5.19)-(5.23) *to obtain* (5.17) *for* $n=1$.

Now, a simple estimate is that $I_n(\tilde\psi)=(\pi/2)\sup|\psi|\delta^{-n}$ so that for $n\geqq 1$

$$I_n(\tilde\psi)\leqq c\sup|\psi|\delta^{-2(n-1)}. \tag{5.24}$$

Next, differentiate $I_{n+1}(\tilde\psi)$ with respect to k_{n+1} and use a trigonometric identity to find that

$$\begin{aligned}
&\frac{d}{dk_{n+1}} I_{n+1}(\tilde\psi, k_1, \cdots, k_{n+1}) \\
&\qquad = \tfrac{1}{2}\{I_n(\tilde{\tilde\psi}, k_1, \cdots, k_n+k_{n+1}) + I_n(\tilde{\tilde\psi}, k_1, \cdots, k_n-k_{n+1})\}
\end{aligned} \tag{5.25}$$

in which $\tilde{\tilde\psi}=\tilde\psi\xi(\sin^2\xi/2+\delta^2)^{-1/2}$ which is again of the form (5.19). Also,

$$I_{n+1}(\tilde\psi, k_1, \cdots, k_n, 0)=0$$

so that

$$\begin{aligned}
&I_{n+1}(\tilde\psi, k_1, \cdots, k_{n+1}) \\
&\qquad = \frac{1}{2}\int_0^{k_{n+1}} [I_n(\tilde{\tilde\psi}, k_1, \cdots, k_n+k_{n+1}) + I_n(\tilde{\tilde\psi}, k_1, \cdots, k_n-k_{n+1})]\, dk_n.
\end{aligned} \tag{5.26}$$

Now, let $k_1, \cdots, k_{n+1}$ be ordered by decreasing magnitude. We have already established (5.17) for $n=1$. By way of induction, we suppose that (5.17) is true for n and show it for $n+1$. If $|k_{n+1}|>\delta^{-2}$, then (5.24) implies (5.17). If $|k_{n+1}|<\delta^{-2}$, then (5.26) and (5.17), with a rearrangement of the k_j's, for n imply that

$$
\begin{aligned}
&|I_{n+1}(\tilde{\psi}, k_1, \cdots, k_{n+1})\\
&\leqq c\left(\prod_{j=2}^{n-1} \phi(k_j, \delta)\right)\frac{1}{2}\int_0^{k_{n+1}} \phi(k_n-k, \delta)+\phi(k_n+k, \delta)\, dk\\
&\leqq c\left(\prod_{j=2}^{n-1} \phi(k_j, \delta)\right)\frac{1}{2}\int_0^{k_{n+1}} [\min(|k_n|-|k|, \delta^{-2})+\min(|k_n|+|k|, \delta^{-2})]\, dk\\
&\leqq c\left(\prod_{j=2}^{n-1} \phi(k_j, \delta)\right)\int_0^{k_{n+1}} \min(|k_n|, \delta^{-2})\, dk \qquad (5.27)\\
&\leqq c\prod_{j=2}^{n-1} \phi(k_j, \delta)|k_{n+1}|\min(|k_n|, \delta^{-2})\\
&\leqq c\prod_{j=2}^{n+1} \phi(k_j, \delta)
\end{aligned}
$$

since $|k_n|>|k_{n+1}|$, which finishes the induction and proves (5.17) after rearrangement of the k_j's. Estimate (5.18) is proved similarly. □

Lemma 5. (Trapezoidal rule for periodic functions.) *If $f(\xi)$ has period 2π and $\sup|f_{\xi\xi}|<c<\infty$ then the error e_T in the trapezoidal rule for integration, i.e.,*

$$e_T=\int_0^{2\pi} f(\xi)\, d\xi-h\sum_1^N f(jh), \tag{5.28}$$

has the bound

$$|e_T|\leqq ch^2\sup|f_{\xi\xi}(\xi)|. \tag{5.29}$$

Proof. Following [1] use the Poisson summation formula, but for $2\pi-$ periodic functions, i.e.,

$$h\sum_1^N f(jh)=2\pi\sum_{m=-\infty}^{\infty}\hat{f}(mN), \tag{5.30}$$

which is easy to prove. Since $\int_0^{2\pi} f(\xi)\, d\xi=2\pi\hat{f}(0)$, then

$$
\begin{aligned}
|e_T|&=2\pi\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}|\hat{f}(mN)|\\
&=2\pi N^{-2}\sum m^{-2}|\hat{f}_{\xi\xi}(mN)| \qquad (5.31)\\
&\leqq 4\pi^2N^{-2}\sup|f_{\xi\xi}|(\sum m^{-2}). \qquad □
\end{aligned}
$$

6. Stability estimates. In this section, the stability error bounds (4.18), (4.19) are derived. To prove (4.18), we require bounds on the Fourier coefficients of the difference $B_\delta[s]-B_\delta[\bar{s}]$. To evaluate and estimate these Fourier coefficients, expand the integral operator B_δ using Lemma 3, with $x=s-s'$, $y=s^*-s^{*\prime}$. Then use the integral bounds in Lemma 4. The expansion is

$$(B_\delta[s]-B_\delta[\tilde{s}])(\gamma, t)$$
$$=\int_{-\pi}^{\pi}[F(s-s', s^*-s^{*\prime}, \xi, \delta)-F(\tilde{s}-\tilde{s}', \tilde{s}^*-\tilde{s}^{*\prime}, \xi, \delta)]\,d\xi$$

$$\begin{aligned}(6.1)\quad =\sum_{N=1}^{\infty}\sum_{\substack{m+n+p+q+r\\=2N-2}}\int_{-\pi}^{\pi}&\cdot[\{a^{(\xi)}_{mnpqr}(s-s'-\tilde{s}+\tilde{s}')+b^{(\xi)}_{mnpqr}(s^*-s^{*\prime}-\tilde{s}^*+\tilde{s}^{*\prime})\}\\ &(s-s')^m(s^*-s^{*\prime})^n(\tilde{s}-\tilde{s}')^p(\tilde{s}^*-\tilde{s}^{*\prime})^q(\sin^2\xi/2)^r(\sin^2\xi/2+\delta^2)^{-N}]\,d\xi,\end{aligned}$$

in which $s=s(\gamma)$, $s'=s(\gamma+\xi)$, etc. The Fourier transform in γ of this is

$$(B_\delta[s]-B_\delta[\tilde{s}])\hat{\ }(k, t)$$
$$\begin{aligned}(6.2)\quad =\sum_{N=1}^{\infty}\sum_{\substack{m+n+p+q+r\\=2N-2}}\sum_{\substack{k_1+\cdots+k_M\\=k}}\prod_{j=1}^{m}\hat{s}_j\prod_{j=m+1}^{m+n}\hat{s}^*_j\prod_{j=m+n+1}^{m+n+p}\hat{\tilde{s}}_j\prod_{j=m+n+p+1}^{m+n+p+q}\hat{\tilde{s}}^*_j\\ \cdot\int_{-\pi}^{\pi}\Bigg[(\sin^2\xi/2)^r\left(\prod_{j=1}^{M}(1-e^{ik_j\xi})\right)(\sin^2\xi/2+\delta^2)^{-N}\\ \cdot\{a^{(\xi)}_{mnpqr}(\hat{s}-\hat{\tilde{s}})(k_M)+b^{(\xi)}_{mnpqr}(\hat{s}^*-\hat{\tilde{s}}^*)(k_M)\}\Bigg]\,d\xi\end{aligned}$$

where $M=m+n+p+q+1$ and $\hat{s}_j=\hat{s}(k_j)$. Now, for $a(\xi)$ that is even in ξ,

$$\begin{aligned}J_N&=\int_{-\pi}^{\pi}\left[a(\xi)(\sin\xi/2)^{2N-M-1}\prod_{j=1}^{M}(1-e^{ik_j\xi})\cdot(\sin^2\xi/2+\delta^2)^{-N}\right]d\xi\\ (6.3)\quad&=(-2i)^M\int_{-\pi}^{\pi}\left[a(\xi)(\sin^2\xi/2+\delta^2)^{-N}e^{ik\xi/2}\prod_{j=1}^{2N-1}\sin\left(\frac{k_j\xi}{2}\right)\right]d\xi\\ &=-(-2i)^{M+1}\int_0^{\pi}\left[a(\xi)(\sin^2\xi/2+\delta^2)^{-N}\prod_{j=1}^{2N}\sin\left(\frac{k_j\xi}{2}\right)\right]d\xi\end{aligned}$$

in which $k_{M+1}=\cdots=k_{2N-1}=1$, $k_{2N}=k$. From Lemma 4, J_N is bounded by

$$\begin{aligned}(6.4)\quad |J_N|&\leqq C^M\sup_{|\xi|\leqq\pi}(|a|+|a'|)\prod_{j=1}^{M}\phi\left(\frac{k_j}{2},\delta\right)\\ &\leqq C^M\sup_{|\xi|\leqq\pi}(|a|+|a'|)\prod_{j=1}^{M}|k_j|.\end{aligned}$$

Using $a=a_{mnpqr}$ or b_{mnpqr} and the bounds from Lemma 3, we estimate $B_\delta[s]-B_\delta[\tilde{s}]$ by

$$|(B_\delta[s]-B_\delta[\tilde{s}])\hat{\ }(k,t)|\leqq\sum_{M=1}^{\infty}\sum_{\substack{k_1+\cdots+k_M\\=k}}\left[C_1^M\left(\prod_{j=1}^{M-1}|k_j\hat{s}_j|\right)|k_M|\,|(\hat{s}-\hat{\tilde{s}})(k_M)|\right]$$

where C_1 is a constant depending on C and a_{mnpqr}, b_{mnpqr},

$$(6.5)\quad \leqq 2\sum_{M=1}^{\infty}\sum_{\substack{k_1+\cdots+k_M\\=k}}C_1^M\left(\prod_{j=1}^{M-1}|(s_\gamma)\hat{\ }(k_j)|\right)|(s-\tilde{s})_\gamma\hat{\ }(k_M)|.$$

Multiply both sides by $e^{\rho|k|}$ and sum over k to obtain

$$\|B_\delta[s]-B_\delta[\tilde{s}]\|_\rho\leqq 2c_1\sup\left(\frac{1}{1-c_1\|s_\gamma\|_\rho},\frac{1}{1-c_1\|\tilde{s}_\gamma\|_\rho}\right)\|s_\gamma-\tilde{s}_\gamma\|_\rho$$

so that

$$\|B_\delta[s]-B_\delta[\tilde{s}]\|_\rho \leqq \kappa\|s_\gamma-\tilde{s}_\gamma\|_\rho. \tag{6.6}$$

Finally (4.18) follows from (6.6) and (5.4). Inequality (4.19) is proved in an analogous fashion.

7. Consistency. The consistency estimates are less delicate than the stability estimates, since they involve known smooth functions. However, to obtain bounds that are uniform in δ, we remove the singularity in B, which leads to a second derivative term in the trapezoidal rule approximation. This term was first noticed by Van de Vooren [15]. As before, evaluation of the Fourier coefficients for estimation by the norms $\|\cdot\|_\rho$ requires power series expansion of the kernels of the integral operators B, B_δ, etc., using Lemma 3. The resulting terms of the series are integrated and bounded using Lemma 4.

First estimate $B-B_\delta$. As in the previous section, the Fourier transform can be written in terms of the trigonometric integral in (5.10). From Lemma 3 this is bounded as

$$\begin{aligned}|(B[s]-B_\delta[s])\hat{}(k)| &= \Bigg| \sum_{N=1}^{\infty} \sum_{\substack{m+n+r\\=2N-1}} \sum_{\substack{k_1+\cdots+k_n\\=k}} \prod_{j=1}^{m} \hat{s}(k_j) \prod_{j=m+1}^{m+n} \hat{s}^*(k_j) \\ &\quad \cdot \int_0^\pi (\sin\xi/2)^r \left(\prod_{j=1}^{m+n} \sin k_j\xi/2\right)\{a_{mn}(\xi,\delta)(\sin^2\xi/2+\delta^2)^{-N} \\ &\qquad - a_{mn}(\xi,0)(\sin\xi/2)^{-2N)}\}\,d\xi \\ &\leqq \delta \sum_{M=1}^{\infty} c^M|k|^2 \prod_{j=1}^{M} |k_j\hat{s}(k_j)|.\end{aligned} \tag{7.1}$$

It follows that for $\|\partial_\gamma s\|_\rho<\kappa$ and $\rho'>\rho>0$,

$$\|B[s]-B_\delta[s]\|_\rho \leqq c\delta(\rho'-\rho)^{-2}\|\partial_\gamma s\|_{\rho'}, \tag{7.2}$$

which with (5.4) implies (4.15).

Next estimate $B-B_p$. In order to apply error estimates for the trapezoidal rule, the singularity in B must be removed. Following Van de Vooren [15] write

$$\begin{aligned} B[s](\gamma) &= (4\pi i)^{-1}\int_{-\pi}^{\pi} \cot\frac{1}{2}(s-s'-\xi)\,d\xi \\ &= (4\pi i)^{-1}\int_{-\pi}^{\pi} \cot\frac{1}{2}(s-s'-\xi)+2(s_\gamma+1)^{-1}\cot\frac{1}{2}\xi\,d\xi,\end{aligned} \tag{7.3}$$

in which $s_\gamma=s_\gamma(\gamma)$ is independent of ξ and the second term in the integral integrates to zero. The integrand in (7.3), which we denote by $f(\gamma,\xi)$, is now analytic in ξ and has the value $f(\gamma,0)=(1+s_\gamma)^{-2}s_{\gamma\gamma}$. According to the trapezoidal rule for periodic functions

$$\begin{aligned} B[s](\gamma) &= h\left\{\frac{1}{2}f\left(\gamma,-\frac{Nh}{2}\right)+f\left(\gamma,\frac{Nh}{2}\right)+\sum_{j=-N/2+1}^{N/2-1} f(\gamma,jh)\right\}+e_T(\gamma) \\ &= h\sum_{j=1}^{N}\cot\frac{1}{2}(s(\gamma)-s(\gamma+jh)-jh)+h(1+s_\gamma(\gamma))^{-2}s_{\gamma\gamma}(\gamma)+e_T(\gamma) \\ &= B_p[s](\gamma)+h(1+s_\gamma)^{-2}s_{\gamma\gamma}+e_T(\gamma).\end{aligned} \tag{7.4}$$

The Van de Vooren term $e_2=h(1+s_\gamma)^{-2}s_{\gamma\gamma}$ is bounded by

$$\|e_2\|_\rho \leqq ch\|s_{\gamma\gamma}\|_\rho \tag{7.5}$$

since $\|s_\gamma\|_\rho \leqq \kappa$.

In order to estimate the ρ-norm of the error $e_T(\gamma)$ for the trapezoidal method, return to the expansion of the kernel of $B[s](\gamma, t)$ given in Lemma 3, i.e.,

$$\begin{aligned} f(\gamma, \xi) = \sum_{m,n=0}^{\infty} a_{mn}(\xi) \frac{(s'-s)^m s_\gamma^n}{(\sin \xi/2)^{m+2}} (s'-s-2s_\gamma \sin \xi/2) \\ + \sum_{m=1}^{\infty} b_m \frac{(s'-s)^m}{(\sin \xi/2)^{m-1}}. \end{aligned} \tag{7.6}$$

We now give the strategy for obtaining this discretization error. By Lemma 5,

$$e_T(\gamma) = \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \hat{f}^{\xi}(\gamma, mN) \tag{7.7}$$

so that

$$\hat{e}_T(k) = \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \hat{f}^{\gamma\,\hat{}\,\xi}(k, mN)$$

where $\hat{}\,\xi$ and $\hat{}\,\gamma$ denote the Fourier transform in ξ and γ, respectively. Thus

$$\begin{aligned} |\hat{e}_T(k)| &\leqq \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} |\hat{f}^{\xi\,\hat{}\,\gamma}(k, mN)| \\ &\leqq 2\pi N^{-2} \sum \frac{1}{m^2} |\hat{f}^{\xi\,\hat{}\,\gamma}_{\xi\xi}(k, mN)| \\ &\leqq 4\pi^2 N^{-2} \sup_{\xi} |\hat{f}^{\gamma}_{\xi\xi}(k, \xi)| \left(\sum \frac{1}{m^2}\right), \end{aligned} \tag{7.8}$$

$$\begin{aligned} \hat{f}(k, \xi) = \sum_{N=1}^{\infty} \sum_{k_1+\cdots+k_N=k} \prod_{j=1}^{N} \hat{s}(k_j) \\ \cdot \left(\sum_{m+n=N-1} a_{mn} \left(\prod_{j=m+2}^{N} |k_j| \right) \prod_{j=2}^{m+1} \left(\frac{e^{ik_j\xi}-1}{\sin \xi/2} \right) \right. \\ \left. \cdot \left(\frac{e^{ik_1\xi}-1-2ik_1 \sin \xi/2}{\sin^2 \xi/2} \right) + b_N \left(\prod_{j=1}^{N} \frac{e^{ik_j\xi}-1}{\sin \xi/2} \right) \sin \xi/2 \right). \end{aligned} \tag{7.9}$$

Then by Lemma 4,

$$|\hat{f}^{\gamma}_{\xi\xi}(k, \xi)| \leqq |k|^2 \sum_{N=1}^{\infty} \sum_{\substack{k_1+\cdots+k_N \\ =k}} |k_N| \prod_{j=1}^{N+1} |k_j| |\hat{s}(k_j)|. \tag{7.10}$$

Combine this with (7.8) to get

$$|\hat{e}_T(k)| \leqq ch^2 |k|^2 \sum_{N=1}^{\infty} \sum_{\substack{k_1+\cdots+k_N \\ =k}} |k_N| \prod_{j=1}^{N+1} |k_j| |\hat{s}(k_j)|. \tag{7.11}$$

Multiplying by $e^{\rho|k|}$ and summing over k, we get

$$\|e_T\|_\rho \leqq ch^2(\rho'-\rho)^{-3} \|s_\gamma\|_{\rho'} \tag{7.12}$$

for $\|s_\gamma\|_{\rho'} < \kappa$ and $\rho' > \rho > 0$. Combining (7.5) and (7.12) gives

$$\|B[s]-B_p[s]\|_\rho \leqq ch(\rho'-\rho)^{-3} \|s_\gamma\|_{\rho'}. \tag{7.13}$$

Now, as in the estimate of $B-B_\delta$, we can show analogously that

$$\|B_D[s]-B_p[s]\|_\rho \leqq c\delta(\rho'-\rho)^{-1}\|s_\gamma\|_{\rho'} \tag{7.14}$$

which together with (7.13) proves (4.16).

Finally, estimate the time discretization error to be

$$(\partial_t - D_t)s((n+1)\Delta t) = \Delta t^{-1}\int_0^{\Delta t} \tau s_{tt}(n\Delta t+\tau)\, d\tau. \tag{7.15}$$

By differentiating (2.3) for s in t, we obtain

$$\partial_t^2 s^* = PV\int (s_t - s_t')\partial_s \cot(s-s'-\xi)/2\, d\xi. \tag{7.16}$$

This can be expanded in powers of $s-s'$ and $s^*-s^{*\prime}$ using Lemma 3, and then the Fourier transform norm $\|\partial_t^2 s\|_\rho$ can be estimated from Lemma 4. Since $\partial_t s^* = B[s]$, the result is

$$\begin{aligned}\|\partial_t^2 s\|_\rho &\leqq \|\partial_\gamma B[s]\|_\rho \\ &\leqq (\rho'-\rho)^{-1}\|\partial_\gamma s\|_{\rho'}.\end{aligned} \tag{7.17}$$

Use (7.17), (7.15), and (5.5) to obtain (4.17). This concludes the proof of Theorem 2. □

Acknowledgment. We thank Robert Krasny for many illuminating conversations and especially for his suggestion that our convergence proof would work for the point vortex approximation by using Van de Vooren's correction term.

Note added in proof. Convergence of the point vortex method for smooth flows has been recently proved by Goodman, Hou, and Lowengrub in $\mathbb{R}^2$ and by Hou and Lowengrub in $\mathbb{R}^3$ (both to appear in Comm. Pure Appl. Math.). Hou and Lowengrub have also shown convergence with spectral accuracy for a modified point vortex method for vortex sheets, which was developed by Shelley to eliminate the Van de Vooren correction term. Their proof requires analytic initial data and exponentially small roundoff error but does not use the discrete Abstract Cauchy-Kowalewski Theorem. As a result it is much simpler than the analysis of this paper.

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