Equilibrium for radiation in a homogeneous plasma

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Entropy increase and comparison principles are found for the Fokker–Planck equation for the radiation field in a homogeneous plasma with constant electron temperature. When emission–absorption is neglected, this is used to find equilibrium distributions that have the form of a Planck distribution plus a δ function at zero photon energy. For distributions below the Planck and with emission–absorption included, a rate of entropy increase is obtained. Numerical results confirm these conclusions.

I. INTRODUCTION

In a fully ionized plasma, the radiation field interacts with the electron and ion fields primarily through Compton scattering with electrons, Bremsstrahlung emission, and inverse Bremsstrahlung absorption. The objective of this paper is to study the approach to equilibrium of the radiation with the matter via these processes. For simplicity, it will be assumed that the plasma is spatially homogeneous, isotropic, and in a thermodynamic equilibrium characterized by a temperature θ, which is artificially held constant in time. While these assumptions are highly idealized, the results obtained should give insight into equilibration under more general conditions.

In the weakly relativistic limit, the energy exchange caused by a single Compton scatter is small, and so the effect of Compton scattering is well described by a nonlinear Fokker–Planck diffusion operator. This diffusion operator was first derived by Kompaneets,[1] and its approximation to the full scattering operator was studied by Cooper.[2] The evolution and equilibration of the radiation distribution is then described by the Fokker–Planck equation. In this paper, two tools are developed for analysis of the equilibration process: an entropy principle and a comparison principle.

We shall consider equilibration of the radiation distribution caused by Compton scattering alone and also that caused by the combined effects of scattering, emission, and absorption. For scattering alone, the entropy principle will be used in Sec. III to show that for some initial data the equilibration must result in Bose condensation. For scattering combined with emission and absorption, precise bounds on the equilibration rate will be found in Sec. IV under some assumptions on the initial state.

II. FOKKER–PLANCK EQUATION

The radiation distribution $f(x,t)$ is defined so that the total photon number density is

$$N(t) = \int_0^\infty f(x,t)x^2\,dx, \tag{1}$$

in which $x = h\nu/\theta$ is the photon energy normalized by the plasma temperature θ. The multiplier x^2 is a geometric factor for spherical symmetry. The nondimensionalized Fokker–Planck equation[1–3] for f is

$$\frac{\partial f}{\partial t} = x^{-2}\left(\frac{\partial}{\partial x}\right)\left[\alpha(x)\left(\frac{\partial f}{\partial x}+f+f^2\right)\right] + \sigma(x)(f_0-f), \tag{2}$$

for $0<x<\infty$ with boundary conditions

$$\alpha(x)\left(\frac{\partial f}{\partial x}+f+f^2\right)=0 \quad \text{at } x=0 \text{ and } x=\infty. \tag{3}$$

The nondimensionalized scattering cross section $\alpha(x)$ and emission–absorption cross section $\sigma(x)$ are both non-negative. The Planck distribution is $f_0(x) = (e^x-1)^{-1}$; it is an equilibrium solution of Eqs. (2) and (3) for any choice of α and σ. The flux of photons in energy space caused by Compton scattering is $F(f,x) = \alpha(x)(\partial f/\partial x + f + f^2)$, and so the boundary conditions [Eq. (3)] state that there is no flux of photons at zero or infinite energies. In the absence of emission and absorption (i.e., $\sigma=0$), the boundary condition [Eq. (3)] implies that an equilibrium state f satisfies $F(f,x)=0$, the solutions of which are the Bose–Einstein distributions f_μ defined by

$$f_\mu(x) = (e^{x+\mu}-1)^{-1} \tag{4}$$

with $\mu \geqslant 0$. The chemical potential μ must be non-negative so that f_μ is positive and nonsingular.

The variation of the entropy S of the photon-plasma system is

$$\delta S = \delta S_{\text{phot}} + \delta S_{\text{plas}}.$$

Assuming the plasma is quasistatic and using conservation of energy, we find that

$$\delta S_{\text{plas}} = (1/\theta)\delta E_{\text{plas}} = -(1/\theta)\delta E_{\text{phot}}.$$

Thus we define the entropy functional $S(f)$ by

$$S(f) = S_{\text{phot}} - (1/\theta)E_{\text{phot}} = \int_0^\infty s(f(x),x)x^2\,dx, \tag{5}$$

where the entropy density $s(f,x)$ is given by

$$s(f,x) = (1+f)\log(1+f) - f\log f - xf. \tag{6}$$

The first two terms in s give the usual Bose–Einstein entropy for S_{phot}, while the xf term gives $(1/\theta)E_{\text{phot}}$ and models the

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entropy exchange between the photons and the plasma.

The entropy density s satisfies

$$\frac{\partial s}{\partial f} = \log\left(1+\frac{1}{f}\right) - x,$$

$$\frac{\partial^2 s}{\partial f^2} = -(f+f^2)^{-1} < 0.$$

Note that $\partial s/\partial f = 0$ at x if $f(x) = f_0(x)$, and that s is concave in f, so that $s(f_0(x),x) \geqslant s(f,x)$ for any f and x, and therefore $s(f_0) \geqslant s(f)$ for any non-negative function. The entropy S is nondecreasing since

$$\frac{dS}{dt} = \int_0^\infty \left(\frac{\partial s}{\partial f}\right)\left(\frac{\partial f}{\partial t}\right) x^2\,dx$$

$$= \int_0^\infty \left[\alpha(x)(f+f^2)^{-1}\left(\frac{\partial f}{\partial x}+f+f^2\right)^2 + \sigma(x)[\log(1+1/f)-x](f_0-f)x^2\right]dx$$

$$\geqslant 0 \qquad (7)$$

for any $\sigma(x) \geqslant 0$ and $\alpha(x) \geqslant 0$, since $[\log(1+1/f) - x]$ and $(f_0 - f)$ have the same sign.

In order to derive a comparison principle for Eq. (2), some assumptions are needed on the behavior of the scattering cross section and on the solution. Suppose that the scattering cross section $\alpha(x)$ satisfies the bounds

$$\begin{aligned} &0 < \alpha(x) < c_1 x^4 \quad \text{for } 0 < x < 1,\\ &0 < \alpha(x) < c_2 x e^x \quad \text{for } 1 < x < \infty. \end{aligned} \qquad (8)$$

These bounds are valid for the Thomson scattering cross section, $\alpha(x) = \overline{\alpha} x^4$, as well as for the more complicated cross section that results from integration of the Kline–Nishina kernel.[2] For simplicity, we also suppose that the solution f of Eqs. (2) and (3) satisfies

$$\begin{aligned} &f(x,t) < c_3(t) x^{-1} \quad \text{for } 0 < x < 1,\\ &\int_1^\infty |f|\,dx < c_4(t). \end{aligned} \qquad (9)$$

These bounds are true of the Planck distribution. However, we have not shown that the solution $f(x,t)$ satisfies Eq. (9) for all t if the initial data $f(x,0)$ does so. In these inequalities, c_1, c_2, c_3, c_4 are bounded and positive. The time dependence of c_3 is essential to allow the solutions described in Sec. III.

Under these assumptions, we shall verify the *Comparison Principle*: *If $f(x,t)$ and $g(x,t)$ are two solutions of Eqs. (2)–(3) both satisfying the bounds (9) and with $f(x,0) \geqslant g(x,0)$ for all x, then $f(x,t) \geqslant g(x,t)$ for all x and t. In particular, if $f(x,0) \geqslant 0$, then $f(x,t) \geqslant 0$.*

This principle is of intrinsic interest and will also be used in the analysis of equilibration rates in Sec. IV.

The normalized difference $h = x^2(f-g)$ satisfies the equation

$$\frac{\partial}{\partial t} h = \frac{\partial^2}{\partial x^2} \alpha x^{-2} h + \frac{\partial}{\partial x} x^{-2}(-\alpha_x + \alpha(1+f+g))h - \sigma h, \qquad (10)$$

for $0 < x < \infty$, with boundary conditions

$$\frac{\partial}{\partial x} h + \left(-\frac{2}{x} + 1 + (f+g)\right) h = 0. \qquad (11)$$

Consider $f(x,t)$ and $g(x,t)$ as known functions, without assuming anything about their relative sizes. Then Eqs. (10) and (11) are forward equations[4,5] for a stochastic diffusion process, with coefficients of diffusion d^2 and drift k given by

$$d^2 = 2x^{-2}\alpha(x),$$

$$k = x^{-2}\alpha_x(x) - x^{-2}(1+f+g)\alpha(x),$$

and with the killing rate being $\sigma(x)$. The standard existence theorem[4] for such a process need only be modified by estimating $f+g$ from above by a function that is not time dependent and satisfies the bounds [Eq. (9)]. As a result, $h(x,t)$ is the probability density for a diffusion process with initial density $h(x,0)$ (which is non-negative), and therefore $h(x,t) \geqslant 0$, which concludes the proof of the comparison principle.

The bounds [Eqs. (8) and (9)] are used in the existence theorem of Ref. 4 to show that the diffusion process never hits the boundaries $x = 0$ or $x = \infty$. This is a technical restriction that can be removed at the expense of more complicated mathematical theory.

III. COMPTON SCATTERING

For Compton scattering alone (i.e., $\sigma = 0$), the Fokker–Planck equation becomes the Kompaneets equation

$$\partial_t f = x^{-2}\,\partial_x[\alpha(x)(\partial_x f + f + f^2)]. \qquad (12)$$

That scattering neither creates nor destroys photons is manifest in the fact that Eq. (12) conserves photon number $N(f) = \int_0^\infty f x^2\,dx$. For the Bose–Einstein distributions, $f_\mu(x) = (e^{x+\mu} - 1)^{-1}$ with $\mu \geqslant 0$, the photon number $N_\mu = N(f_\mu)$ is a finite monotone decreasing function of μ, $N_\mu \leqslant N_0$. If an initial photon distribution f_* has $N(f_*) \leqslant N_0$, then when the entropy $S(f)$ is maximized subject to the constraint that $N(f) = N(f_*)$, it is found that the maximum is attained when $f = f_\mu$, where μ is determined by $N(f_\mu) = N(f_*)$. The parameter μ enters the analysis as a Lagrange multiplier. Thus it is reasonable to conclude that the resulting solution of Eq. (12) equilibrates to this f_μ.

If, on the other hand, an initial photon distribution f_* has $N(f_*) > N_0$, then when the entropy $S(f)$ is maximized subject to the constraint that $N(f) = N(f_*)$, it is found that the maximum is never attained within the class of positive regular distribution functions (i.e., without point masses). We show below that $S(f)$ can take values arbitrarily close to $S(f_0)$ and that as $S(f) \to S(f_0), f \to f_0$ away from $x = 0$. Thus it is reasonable to conclude that the resulting solution of Eq. (12) equilibrates to f_0 away from $x = 0$, and piles the excess photons near $x = 0$. This is the so-called Bose condensation.

Numerical calculations presented in Figs. 1 and 2 illustrate the Bose condensation behavior. The initial datum is $f(x,0) = (e^{x/2} - 1)^{-1}$, a Planckian at double the temperature with $N(f) = 8N_0$. The Kompaneets equation [Eq. (12)] is solved with the classical diffusion coefficient $\alpha = x^4$ and no emission–absorption ($\sigma = 0$) by the finite difference

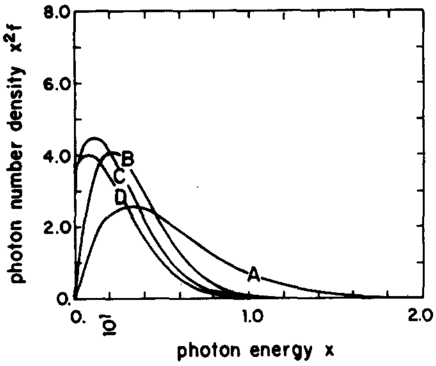

FIG. 1. Radiation number density x^2f versus photon energy x at uniform time intervals. Curve A is the initial Planckian at temperature 2. Curves B, C, D, at successive time intervals, show the shift of the photons towards zero energy. Spikes near $x = 0$ for curves C and D are too narrow to be resolved on this scale and exceed the range of the graph (see Fig. 2).

scheme developed by Larsen, Levermore, Pomraning, and Sanderson.[6]

Analysis of Bose condensation is performed by finding the state of maximum entropy. It is shown below that

$$\text{if} \int_{x>0} |f-f_0|x^2\,dx > 0, \quad \text{then } S(f) < S(f_0), \tag{13}$$

$$\text{if} \quad f = f_0 + n\delta(x), \quad \text{then } S(f) = S(f_0). \tag{14}$$

To verify Eqs. (13) and (14), first note that any positive distribution function f with finite number N [defined by Eq. (1)] consists of a regular function plus a sum of δ functions, i.e.,

$$f(x) = \tilde{f}(x) + \sum_{i=0}^{\infty} n_i\, x_i^{-2}\delta(x - x_i), \tag{15}$$

in which $\tilde{f}$ is a regular function and $n_i > 0$ with $\Sigma n_i < \infty$.

Next evaluate the entropy $S(x_i^{-2}\,\delta(x - x_i))$ for a δ function by considering a sequence of functions $g_k(x)$ approaching $x_i^{-2}\,\delta(x - x_i)$ as k approaches ∞. The limit is taken to mean that $g_k \geqslant 0$ and $\int_0^\infty u(x)g_k(x)x^2\,dx \to u(x_i)$ for any u, with the additional technical condition that $\int_{|x-x_i|>\epsilon} g_k(1 + |\log g_k|)x^2\,dx \to 0$ as $k \to \infty$ for any $\epsilon > 0$.

First it will be shown that the terms $(1+f)\log(1+f) - f\log f$ in the entropy are insignificant for such functions g_k that approximate a δ function. Since

$$|(1+g)\log(1+g)| < cg(1 + |\log g|),$$

and

$$|(1+g)\log(1+g) - g\log g| \leqslant cg^{1/2},$$

for some constant c independent of $g \geqslant 0$, then

$$\begin{aligned}
\int_0^\infty &[(1+g_k)\log(1+g_k) - g_k\log g_k]x^2\,dx \\
&\leqslant c\int_{|x-x_i|>\epsilon} g_k(1 + |\log g_k|)x^2\,dx \\
&\quad + c\int_{|x-x_i|<\epsilon} g_k^{1/2}x^2\,dx \\
&\leqslant c\int_{|x-x_i|>\epsilon} g_k(1 + |\log g_k|)x^2\,dx \\
&\quad + c\epsilon^{1/2}\left(\int_{|x-x_i|<\epsilon} g_k x^4\,dx\right)^{1/2},
\end{aligned}$$

using the Cauchy–Schwartz inequality in the last estimate. Both terms on the right side of this estimate approach zero if first k is taken to ∞, and then ϵ to zero. Then it follows from the definition [Eqs. (5), (6)] for S that for k large, $S(g_k) \simeq \int_0^\infty -x^3\,g_k(x)dx$ and therefore

$$\begin{aligned}
S(nx_i^{-2}\delta(x - x_i)) &= \int_0^\infty -xn\delta(x - x_i)dx \\
&= -x_i\,n.
\end{aligned} \tag{16}$$

Now the entropy density $s(f(x),x)$ is just a function of the values $f(x)$ and x, and thus for f as in Eq. (15),

$$\begin{aligned}
S(f) &= S(\tilde{f}) + \sum_{i=0}^{\infty} S(n_i x_i^{-2}\delta(x - x_i)) \\
&= S(\tilde{f}) - \sum_{i=0}^{\infty} n_i\,x_i.
\end{aligned} \tag{17}$$

Moreover, $s(f,x)$ has its maximum at $f = f_0(x)$, so

$$s(\tilde{f}(x),x) < s(f_0(x),x) \quad \text{if} \quad \tilde{f}(x) \neq f_0(x). \tag{18}$$

From Eqs. (17) and (18), it follows that $S(f) < S(f_0)$, unless $\tilde{f} = f_0$ and the only δ function in f is $\delta(0)$, i.e., $x_i = 0$. This proves Eqs. (13) and (14).

Since the entropy $S(f)$ increases as f evolves through the Kompaneets equation [Eq. (12)], the equilibrium state f is the distribution that maximizes S subject to the constraint that the photon number $N(f)$ is fixed. According to Eqs. (13) and (14), S is maximized for $f = f_0 + nx^{-2}\delta$. Thus we have shown that for solutions f of Eq. (12) with $N(f) > N_0$, the limit $f(x,t)$ as $t \to \infty$ is given by

$$f(x) = f_0(x) + nx^{-2}\delta(x), \tag{19}$$

in which $n = N(f) - N_0$ and $\delta(x)$ is the Dirac delta function.

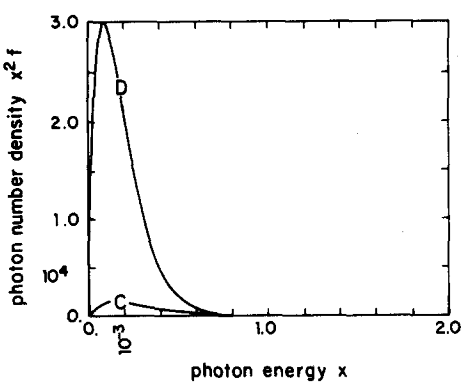

FIG. 2. The spikes in x^2f for curves C and D on a fine x scale near $x = 0$. The scale for x^2f in Fig. 2 is 10^4 times that in Fig. 1. The development of the δ function in x^2f is seen.

IV. SCATTERING, EMISSION, AND ABSORPTION

Relevant choices for the emission–absorption rate $\sigma(x)$ are

$$\sigma(x) = \bar{\sigma}x^{-3}(1 - e^{-x}), \tag{20}$$

$$\sigma(x) = \bar{\sigma} x^{-n}(1 - e^{-x}), \tag{21}$$

$$\sigma(x) = \bar{\sigma} x^{-3} e^{-x/2} K_0(x/2). \tag{22}$$

The last expression (22) is that found through the Born approximation in which K_0 is the zero-order modified Bessel function; the other two [Eqs. (20), (21)] are simple approximations. Since there is no conserved quantity for the Fokker–Planck equation [Eq. (2)], there is only one equilibrium—the Planck distribution $f_0(x) = (e^x - 1)^{-1}$. This agrees with the principle of detailed balance, which says that in equilibrium, both scattering and emission absorption terms must vanish separately. Also from Eq. (7), the entropy increase rate dS/dt vanishes only if $[\log(1 + 1/f) - x](f_0 - f) = 0$, which again requires that $f = f_0$. No singularity δ function is allowed at $x = 0$ because of the singularity of σ at $x = 0$.

Explicit estimates of the rate at which f approaches f_0 may be made in the special case that $f \leqslant f_0$ for all x, so that $s(f, x) > 0$. Since f_0 is an equilibrium, the comparison principle of Sec. I says that $f \leqslant f_0$ for all t if it is true at $t = 0$. Denote $f = (1 - h) f_0$, in which $1 \geqslant h \geqslant 0$, and also define

$$\begin{aligned} q(f,x) &= [\log(1 + 1/f) - x](f_0 - f) \\ &= h f_0 \log[(1 - e^{-x} h)/(1 - h)], \end{aligned} \tag{23}$$

which appears in the integrand in Eq. (7). Then

$$s(f_0(x),x) = \log[e^x/(e^x - 1)], \tag{24}$$

$$\begin{aligned} &s(f_0(x),x) - s((1 - h) f_0(x),x) \\ &\quad = f_0((1 - h)\log(1 - h) - (e^x - h)\log(1 - e^{-x} h)). \end{aligned} \tag{25}$$

For $y > -1$, $(d/dy)\log(1 + y) > (d/dy)(1 + 1/y) \times \log(1 + y) > 0$. It follows that for any y_1 and y_2 with $y_1 > -1$ and $y_2 > -1$,

$$0 < \frac{(1 + 1/y_1)\log(1 + y_1) - (1 + 1/y_2)\log(1 + y_2)}{\log(1 + y_1) - \log(1 + y_2)} < 1. \tag{26}$$

Set $y_1 = -h$ and $y_2 = -h e^{-x}$ in Eq. (26) and use Eqs. (23) and (25) to obtain

$$0 < \frac{s(f_0,x) - s((1 - h) f_0,x)}{q((1 - h) f_0,x)} < 1$$

for any $h < 1$. Since $s(f_0,x) \geqslant s((1 - h) f_0,x)$, this shows that

$$s(f_0,x) - s((1 - h) f_0,x) < q((1 - h) f_0,x). \tag{27}$$

Denote $\Delta S = S(f_0) - S((1 - h) f_0)$. We estimate the approach of ΔS to 0, assuming that $\sigma(x)$ is a decreasing function of x [as in Eqs. (20)–(22)]. For any choice of $\bar{x}(t)$ (to be optimized later), use Eqs. (7) and (27) to estimate

$$\begin{aligned} \frac{d\Delta S}{dt} &\leqslant -\int_0^\infty \sigma(x) q((1 - h) f_0(x),x) x^2\, dx \\ &\leqslant -\sigma(\bar{x}(t)) \int_0^{\bar{x}(t)} [s(f_0(x),x) \\ &\qquad - s((1 - h) f_0(x),x)] x^2\, dx \\ &\leqslant -\sigma(\bar{x})\Delta S + \sigma(\bar{x}) \int_{\bar{x}}^\infty s(f_0,x) x^2\, dx \\ &\leqslant -\sigma(\bar{x})\Delta S + 2\sigma(\bar{x})\bar{x}^2 e^{-\bar{x}}. \end{aligned} \tag{28}$$

Using Eq. (20) for $\sigma(x)$ and choosing $\bar{x} = \lambda t^\gamma$, Eq. (28) becomes

$$\frac{d\Delta S}{dt} \leqslant -\bar{\sigma}\lambda^{-3} t^{-3\gamma} \Delta S + \bar{\sigma} 2\lambda^{-1} t^{-\gamma} e^{-\lambda t^\gamma},$$

$$\begin{aligned} \Delta S(t) \leqslant \Delta S(0) &\exp[-\bar{\sigma} t^{1-3\gamma}/\lambda^3(1 - 3\gamma)] \\ &+ 2\bar{\sigma}\lambda^{-1} \int_0^t r^{-\gamma} \exp[-\lambda r^\gamma \\ &- \bar{\sigma}(t^{1-3\gamma} - r^{1-3\gamma})/\lambda^3(1 - 3\gamma)] dr. \end{aligned}$$

The choice of $\gamma = 1/4$, $\lambda = 2\bar{\sigma}^{-1/4}$ leads to

$$0 \leqslant \Delta S(t) \leqslant [\Delta S(0) + c] \exp[-(\bar{\sigma} t)^{1/4}/4], \tag{29}$$

in which $c = \bar{\sigma}^{3/4} \int_0^\infty r^{-1/4} e^{-3r^{1/4}/2} dr = 64\bar{\sigma}^{3/4}/27$.

The final result [Eq. (29)] shows the rate of decrease of ΔS and therefore gives a measure of the speed at which f converges to f_0. Similar rates can be established for other choices of σ.

V. CONCLUSIONS

Entropy increase and comparison principles have been derived for the Fokker–Planck equation, which describes the radiation distribution in a homogeneous plasma. The entropy function is used to find the equilibrium distributions for scattering alone and for scattering with emission absorption. Besides the usual Planck and Bose–Einstein equilibrium, for scattering alone there are additional equilibria in the form of a Planckian plus a δ-function distribution at zero energy. For systems obeying Bose–Einstein statistics, Bose condensation has been predicted by thermodynamic arguments.[7] Indeed, in the absence of emission and absorption, the photon number is conserved and the photons may be thought of as bosons with a nonzero chemical potential, and so the occurrence of Bose condensation is natural. Our analytic derivation and numerical results confirm that the Fokker–Planck equation describes the Bose condensation (and its development in time) for the photon distribution. The behavior is significant in physical situations[8,9] for which emission and absorption are weak at all but the lowest energies.

When emission–absorption is included, the rate of approach of entropy $S(f)$ to its equilibrium value has been found, at least for distributions f that are below the equilibrium f_0. This is one of the few examples in statistical physics for which such a rate can be derived.

The results here will be used in two subsequent papers that present a solution of the Fokker–Planck equation with small emission absorption. The solution will be found approximately as a Bose–Einstein distribution f_μ, with μ slowly decreasing to the equilibrium value 0.

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