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Wave propagation in bubbly liquids at finite volume fraction

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We derive effective equations for wave propagation in a bubbly liquid in a linearized low-frequency regime by a multiple-scale method. The effective equations are valid for finite volume fraction. For periodic bubble configurations, effective equations uniformly valid for small volume fraction are obtained. We compare the results to the ones obtained in a previous paper (Caflisch *et al.* 1985) for a nonlinear theory at small volume fraction.

1. Introduction

In this paper we derive effective (or macroscopic) equations for sound propagation in a bubbly liquid at finite, but not small, volume fraction in the linearized (amplitude much smaller than the bubble size), low-frequency regime. As the volume fraction becomes small we will find that we reproduce the effective wave speed predicted by Crespo (1969). Therefore our result accounts for the relative drift and deformations of the bubbles.

In a previous paper (Caflisch *et al.* 1985, hereinafter referred to as I) we showed how a system of nonlinear equations proposed by Van Wijngaarden (1968, 1972) for the analysis of sound propagation in a bubbly liquid can be derived from a microscopic formulation. There we used Foldy's method (Foldy 1945; Carstensen & Foldy 1947) in a nonlinear context. A consequence of that analysis is that the effective equations of Van Wijngaarden were shown to be valid for mixtures with very small gas bubble volume fraction.

We shall rederive directly effective equations in the linearized regime and then compare them with the linearized results of I to emphasize several differences. In particular, in I the dominant mode of bubble oscillation is radial and there is no net drift of the bubbles to leading order. On the other hand, at larger volume fraction the dominant mode of oscillation is non-radial and there is a drift between the bubbles and the liquid. In addition, the gas now appears to be incompressible to leading order. Because we are considering only low frequencies and since the liquid is nearly incompressible we see that at finite volume fractions the incompressibility of the gas is a consequence of conservation of local liquid volume.

Although the *microscopic* problem is incompressible at leading order, at second order compressibility effects enter. Furthermore the leading-order *macroscopic* equations are affected by the compressibility. This phenomenon of coupling of scales is important and occurs in most multiple-scale methods. Additional discussion of this compressibility effect is found in the remarks following (3.16).

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In §2 we formulate the problem at the microscopic level. We rely here on I to avoid repetition of details but our discussion is self-contained. The macroscopic equations are derived in §3 by a multiple-scale method for general random configurations of gas bubbles. Finite-volume-fraction effects are discussed in that section.

In §4 we show how, in the special case of a periodic configuration of gas bubbles, we can obtain effective equations valid both for small and finite volume fraction in a uniform way. We can then compare directly the results of §3 and those of our previous paper.

2. The linearized microscopic problem

The description of the problem and the relevant scaling is given in detail in §§2 and 3 of I. We will repeat and generalize here several of the details and summarize the necessary facts. Then we will give the equations in dimensionless form.

We are interested in wave-propagation phenomena in a gas–liquid mixture. Consider a collection of N gas bubbles centred at $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ and surrounded by a liquid. Let λ be the wavelength of a disturbance propagating in the bubbly liquid, V the volume of the region Ω containing the bubbles and R_0 a typical bubble radius. There are two basic dimensionless parameters: the dimensionless interbubble centre distance

$$\epsilon = \frac{1}{\lambda}\left(\frac{V}{N}\right)^{\frac{1}{3}}, \tag{2.1}$$

and the dimensionless bubble radius

$$\delta = \frac{R_0}{\lambda}. \tag{2.2}$$

The gas volume fraction β is given by

$$\beta = \frac{1}{V}\tfrac{4}{3}\pi R_0^3 N = \tfrac{4}{3}\pi\left(\frac{\delta}{\epsilon}\right)^3. \tag{2.3}$$

Let p_0 be a typical pressure, say the equilibrium pressure, in the bubbly liquid. The resonant bubble frequency (cf. (5.9) in I) for spherical oscillations of a typical bubble is

$$\omega_0 = \left(\frac{3\gamma p_0}{R_0^2 \rho_\ell}\right)^{\frac{1}{2}}, \tag{2.4}$$

where γ is the ratio of specific heats of the gas and ρ_ℓ the density of the liquid, assumed constant. We will consider only frequencies ω such that $\omega \ll \omega_0$. In addition we define two other dimensionless parameters (cf. (3.8) and (3.9) in I)

$$\zeta = \frac{1}{\delta^2}\frac{p_0}{\rho_\ell \bar{c}^2}, \tag{2.5}$$

$$C^2 = (c_\ell/\bar{c})^2. \tag{2.6}$$

Here c_ℓ is the liquid sound speed (assumed constant) and $\bar{c}$, defined by (3.22), is the effective sound speed of the bubbly liquid.

We will write down the equations only in dimensionless form. Therefore let λ, f, ρ_ℓ, R_0, p_0, $\bar{c}\delta^2$ be the reference length, frequency, density, bubble radius, pressure and velocity. Here we set $\lambda f = \bar{c}$. Also let the surface of each bubble be given by $|\boldsymbol{x}-\boldsymbol{x}_j| = \delta R_j(t)$. We have suppressed the dependence of the surface, R_j, on the local

angular coordinates. Note that for a spherically symmetric bubble the radius R_j depends only on t. The non-dimensionalized equations of motion in the liquid region $\{\boldsymbol{x}: |\boldsymbol{x}-\boldsymbol{x}_j| > \delta R_j \text{ for all } j\}$ are

$$\frac{\zeta}{C^2}(p_t+\delta^2\boldsymbol{u}\cdot\nabla p)+\nabla\cdot\boldsymbol{u} = 0, \tag{2.7}$$

$$\boldsymbol{u}_t+\delta^2\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\zeta\nabla p = 0. \tag{2.8}$$

Inside each gas bubble we have the equations

$$\frac{1}{\gamma p}(p_t+\delta^2\boldsymbol{u}\cdot\nabla p)+\delta^2\nabla\cdot\boldsymbol{u} = 0, \tag{2.9}$$

$$\frac{\rho}{\rho_l}(\boldsymbol{u}_t+\delta^2\boldsymbol{u}\cdot\nabla\boldsymbol{u})+\zeta\nabla p = 0. \tag{2.10}$$

Here p and $\boldsymbol{u}$ are the scaled pressure and velocity fields in the gas or in the liquid and ρ is the density in the gas.

On the surface of each bubble $|\boldsymbol{x}-\boldsymbol{x}_j| = \delta R_j(t)$ we have the interface conditions

$$\boldsymbol{u}\cdot\boldsymbol{n} = \text{continuous}, \quad p \text{ continuous}, \tag{2.11}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = \frac{1}{\delta}\frac{\partial}{\partial t}(R_j[1+|\nabla R_j|^2/R_j^2]^{-\frac{1}{2}}), \tag{2.12}$$

where $\boldsymbol{n}$ is the unit normal to the bubble surface and ∇ is the angular gradient. In (2.11) we have for simplicity ignored surface-tension effects. In the gas we have used the dimensionless isentropic equation of state

$$p = \left(\frac{\rho}{\rho_g}\right)^\gamma \quad \text{with} \quad \rho_g = \frac{M_0}{\frac{4}{3}\pi R_0^3}. \tag{2.13}$$

Here ρ_g is the density of the gas at p_0. We assume that the bubbles at the equilibrium pressure p_0 are spherical with the same radius R_0 and contain the same mass of gas M_0. Initial conditions for p, $\boldsymbol{u}$ and R_j are also given.

In (2.7)–(2.13) we have made no assumption concerning the shape of the bubbles, leaving it to be determined. In I it was assumed that the bubbles were all spherical and that there was no drift relative to the fluid. This is the correct assumption concerning the leading-order behaviour of the bubbles under the limit considered in I, i.e.

$$\epsilon\to 0, \quad \delta\to 0, \quad \chi = \frac{\delta}{\epsilon^3} \text{ fixed}. \tag{2.14}$$

Suppose we consider the limiting case (2.14) of I. Since the ratio ρ_g/ρ_ℓ is typically small, of order 10^{-3} for air and water, this suggests that the pressure distribution inside each bubble tends to become spatially uniform. A formal expansion in powers of ρ_g/ρ_ℓ for the problem (2.7)–(2.13) gives a set of simplified equations. They are (2.7) and (2.8) in the fluid with the boundary condition (2.12) on the spherical bubbles. In the gas phase (2.9) and (2.10) are solved by prescribing uniform pressure p and density ρ in the bubble with p given by (2.13), and by using the boundary condition (2.11), i.e. the continuity of p.

As will be seen more explicitly in this paper, this simplification of (2.7)–(2.13) for small ρ_g/ρ_ℓ is valid only when the volume fraction β defined by (2.3) is small, as in the limit (2.14) in which $\beta = O(\delta^2)$. In this case of small gas volume fraction, pressure gradients inside the bubbles are negligible.

When the gas volume fraction is not small we shall show that pressure gradients inside the bubbles cannot be ignored even though ρ_g/ρ_ℓ is small. Therefore we must analyse the full system (2.7)–(2.13). We shall do this in the linear regime only, i.e. around the equilibrium state $p = 1$, $\boldsymbol{u} = 0$ in dimensionless variables when the amplitude of oscillation is much smaller than δ. The linearized dimensionless equations of motion are

$$\frac{\zeta}{C^2} p_t + \nabla \cdot \boldsymbol{u} = 0, \tag{2.15}$$

$$u_t + \zeta \nabla p = 0 \tag{2.16}$$

in the liquid region $\{\boldsymbol{x} : |\boldsymbol{x} - \boldsymbol{x}_j| > \delta\}$ (note that we have linearized about the equilibrium radius δ), and

$$\frac{1}{\gamma\delta^2} p_t + \nabla \cdot \boldsymbol{u} = 0, \tag{2.17}$$

$$\tau \boldsymbol{u}_t + \zeta \nabla p = 0 \tag{2.18}$$

in the gas. Here τ is defined by

$$\tau = \frac{\rho_g}{\rho_\ell}. \tag{2.19}$$

At the gas-bubble surfaces $|\boldsymbol{x} - \boldsymbol{x}_j| = \delta, j = 1, \ldots, N$, we have the interface conditions

$$p \text{ and } \boldsymbol{u} \cdot \boldsymbol{n} \text{ continuous.} \tag{2.20}$$

Eliminating $\boldsymbol{u}$, (2.15)–(2.18) can be written as a single equation

$$A p_{tt} + \nabla \cdot (B \nabla p) = 0, \tag{2.21}$$

both in the liquid and in the gas, where

$$A = \begin{cases} C^{-2} & \text{in liquid,} \\ (\gamma\zeta\delta^2)^{-1} & \text{in gas,} \end{cases} \tag{2.22}$$

$$B = \begin{cases} 1 & \text{in liquid,} \\ \tau^{-1} & \text{in gas.} \end{cases} \tag{2.23}$$

The interface conditions are now

$$p \text{ and } \boldsymbol{n} \cdot B \nabla p \quad \text{continuous} \tag{2.24}$$

across the bubble surfaces $|\boldsymbol{x} - \boldsymbol{x}_j| = \delta$.

Before carrying out the analysis for the finite volume fraction, we point out the connection of (2.21)–(2.24) with our previous paper I. Equation (2.21) can be analysed in the continuum limit (2.14) where the gas bubble centres satisfy the condition (cf. (3.15) in I)

$$\frac{V}{\lambda^3} \frac{1}{N} (\text{number of } \boldsymbol{x}_j \text{ in any set } A) \to \int_A \theta(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}. \tag{2.25}$$

The function $\theta(x)$ is the scaled (by the volume V) continuum bubble-centre density. As in I, we can apply Foldy's method to this problem in the limit (2.14) to derive an effective linear equation. When we also expand in $\tau \to 0$ we obtain the effective equation

$$(C^{-2} \partial_t - \Delta)(\partial_t^2 + 3\gamma\zeta)\bar{p} + 4\pi\theta\chi\bar{p}_{tt} = 0, \tag{2.26}$$

which is precisely the linearized version of the effective equations found in I. Equation (2.26) is used frequently to analyse many propagation phenomena in bubbly liquids.

It is interesting to note the following.

(i) Although the pressure is uniform within a bubble, it varies through the fluid. On a microscopic scale pressure variations correspond to a monopole and give radial velocity fields in both phases.

(ii) The limit (2.14), which is the same as $\delta \to 0$, $N \to \infty$ with χ, ζ and C fixed, and the limit $\tau \to 0$ are interchangeable. That is, the Foldy approximation and the small-gas-bubble-inertia approximation can be performed in any order.

(iii) The Foldy approximation and the linearization (small-amplitude approximation) can be carried out in any order.

All these approximations commute because in the Foldy approximation each bubble feels only the macroscopic pressure field about it and not the local fields of the other bubbles. This of course will not be true at finite gas volume fractions.

3. Analysis at larger volume fractions

We want to analyse (2.15)–(2.18), or equivalently (2.21), when β is not going to zero. In contrast with (2.14), we consider the limit

$$\delta \to 0, \quad \epsilon \to 0, \quad \text{with } \beta = \tfrac{4}{3}\pi\left(\frac{\delta}{\epsilon}\right)^3 \text{ fixed}, \tag{3.1}$$

i.e. here we assume $\beta = O(1)$ relative to ϵ, whereas in (2.14) we assumed $\beta = O(\epsilon^6)$. We also want to consider low frequencies, that is, $\omega \sim \delta\omega_0$. Because of the definition (2.4) and (2.5), this condition corresponds to requiring that

$$\zeta^* = \zeta\delta^2 \text{ stays fixed as } \epsilon \text{ and } \delta \text{ tend to zero}; \tag{3.2}$$

where $\zeta^* = p_0/\rho_\ell \bar{c}^2$. Our intent is to apply the method of multiple scales (see e.g. Keller 1977, 1980; Bensoussan, Lions & Papanicolaou 1978; Sanchez-Palencia 1980) to derive an effective set of equations for wave propagation in a bubbly liquid. Although here we will only give a formal derivation of the equations, a rigorous justification of our result can be obtained. This was done on a similar problem by Papanicolaou & Varadhan (1981). Several technical assumptions are needed to justify our formal methods, and, although we will not present the details, we will list some assumptions and hypotheses that are needed.

The hypotheses about the bubble centres $\{\boldsymbol{x}_j\}$ are needed because the effective equations will not depend only on the average density of $\{\boldsymbol{x}_j\}$, as in the case of the limit (2.14) where the bubble volume fraction goes to zero, but also on statistics of the interbubble distances. We shall assume now that the bubble centres $\{\boldsymbol{x}_j\}$ form a random distribution of infinitely many points in space. The distribution is assumed to be stationary in the sense that the joint probability distribution of these points is translation invariant. We shall denote averaging or expectation by $\langle\ \rangle$ and assume that the average bubble-centre density equals ϵ^{-3}, which conforms with our previous scaling. If for example ψ is a function that vanishes outside a bounded region then

$$\langle \sum_j \psi(\boldsymbol{x}_j) \rangle = \epsilon^{-3} \int_{\mathbb{R}^3} \psi(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}.$$

Note that we have made no assumptions about statistical independence of the bubble centres – just stationarity. To apply the method of multiple scales it is convenient to introduce a simplifying assumption regarding the dependence of the point process $\{\boldsymbol{x}_j\}$ on ϵ. We shall assume that there is a point process $\{\boldsymbol{y}_j\}$ that is stationary, does not depend on ϵ, has average intensity (average number of points per unit volume) equal to one and such that $\{\boldsymbol{x}_j\}$ and $\{\epsilon \boldsymbol{y}_j\}$ are statistically equivalent.

We must assume that the distribution of bubble centres is such that overlapping of bubbles is not permitted. So we assume that the point process $\{\boldsymbol{y}_j\}$ satisfies the constraint $|\boldsymbol{y}_i - \boldsymbol{y}_j| > 2\xi$, with probability one for all i different from j, where

$$\xi = \delta/\epsilon \tag{3.3}$$

is the ratio of the bubble radius to the mean interbubble distance.

In view of (2.22) and (2.23), we define two stationary random functions $A(\boldsymbol{y})$ and $B(\boldsymbol{y})$ as follows:

$$A(\boldsymbol{y}) = A(\boldsymbol{y}, \{\boldsymbol{y}_j\}) = \begin{cases} C^{-2} & \text{if } |\boldsymbol{y}-\boldsymbol{y}_j| > \xi \quad \text{for all } j, \\ (\gamma\zeta\delta^2)^{-1} & \text{if } |\boldsymbol{y}-\boldsymbol{y}_j| \leqslant \xi \quad \text{for some } j, \end{cases} \tag{3.4}$$

and

$$B(\boldsymbol{y}) = B(\boldsymbol{y}, \{\boldsymbol{y}_j\}) = \begin{cases} 1 & \text{if } |\boldsymbol{y}-\boldsymbol{y}_j| > \xi \quad \text{for all } j, \\ \tau^{-1} & \text{if } |\boldsymbol{y}-\boldsymbol{y}_j| > \xi \quad \text{for some } j, \end{cases} \tag{3.5}$$

where $\boldsymbol{y} = \boldsymbol{x}/\epsilon$. Then (2.21) can be written in the form

$$A\left(\frac{\boldsymbol{x}}{\epsilon}\right) p_{tt} - \nabla \cdot \left(B\left(\frac{\boldsymbol{x}}{\epsilon}\right) \nabla p\right) = 0, \tag{3.6}$$

with the same interface conditions (2.24). We do not write explicitly the dependence on the bubble configuration $\{\boldsymbol{y}_j\}$, but of course $p = p(t, \boldsymbol{x}, \{\boldsymbol{y}_j\}, \epsilon)$ in (3.6). Initial conditions for (3.6) are

$$p = \tilde{p}(\boldsymbol{x}), \quad p_t = \tilde{q}(\boldsymbol{x}) \quad \text{at } t = 0, \tag{3.7}$$

where $\tilde{p}(\boldsymbol{x})$ and $\tilde{q}(\boldsymbol{x})$ are deterministic smooth functions.

The analysis of (3.6), (3.7) is carried out by a multiscaling procedure (Papanicolaou & Varadhan 1981, and references therein), in which the stationarity of the bubble centres $\{\boldsymbol{y}_j\}$ plays an important role.

We shall show that the effective equation associated with (3.6) has the form

$$\langle A \rangle \bar{p}_{tt} - \sum_{i,j=1}^{3} q_{ij}(\beta, \tau) \frac{\partial^2 \bar{p}}{\partial x_i \partial x_j} = 0 \tag{3.8}$$

with initial conditions (3.7). From (3.4) the average of A is readily calculated:

$$\langle A \rangle = \frac{1-\beta}{c_\ell^2} + \frac{\beta}{\gamma\zeta\delta^2}, \quad \beta = \tfrac{4}{3}\pi\xi^3. \tag{3.9}$$

Recall that in the limit (3.1) $\zeta\delta^2$ stays fixed, as does $\xi = \delta/\epsilon$ and hence the volume fraction $\beta = \frac{4}{3}\pi(\delta/\epsilon)^3$. The symmetric tensor q_{ij} is defined by (3.16) below. Unlike $\langle A \rangle$ the values of q_{ij} depend on the bubble configuration and cannot be calculated exactly, even in the periodic case. For small β one can obtain an approximation for q_{ij}. This and several other aspects of (3.8) are discussed after its derivation.

Now we proceed with the derivation of (3.8). For simplicity of presentation we shall give the details here with a periodic distribution of points $\{\boldsymbol{y}_j\}$ (of period one), but

we shall state all results in general. Now $\langle\ \rangle$ will denote spatial averaging over a single period (or cell). We look for the solution p of (3.6) in the form

$$p(t, \boldsymbol{x}, \{\boldsymbol{y}_j\}, \epsilon) = \bar{p}(t, \boldsymbol{x}) + \epsilon p_1(t, \boldsymbol{x}, \boldsymbol{y}, \{\boldsymbol{y}_j\}) + \epsilon^2 p_2(t, \boldsymbol{x}, \boldsymbol{y}, \{\boldsymbol{y}_j\}) + \dots, \tag{3.10}$$

where dependence on $\{\boldsymbol{y}_j\}$ (the periodic array of bubble centres) will be omitted in the sequel. This *Ansatz* is motivated by the form of the coefficients in (3.6), which are periodic functions of $\boldsymbol{y}$. We use the fact that

$$\nabla p = [\nabla_x + \epsilon^{-1}\nabla_y]\, p(t, \boldsymbol{x}, \boldsymbol{y})|_{\boldsymbol{y} = \boldsymbol{x}/\epsilon},$$

insert (3.9) in (3.6) and collect coefficients of powers of ϵ. There are no terms of order ϵ^{-2} since $\bar{p}$ has been assumed to be independent of $\boldsymbol{y}$. The terms of order ϵ^{-1} give

$$\nabla_y \cdot (B(\boldsymbol{y})\, \nabla_y p_1) + \nabla_y \cdot (B(\boldsymbol{y})\, \nabla_x \bar{p}) = 0, \tag{3.11a}$$

which is an equation for p_1 as a function of $\boldsymbol{y}$ with t and $\boldsymbol{x}$ as parameters. Let $\boldsymbol{e}_k$ be the unit vector in the kth direction, $k = 1, 2, 3$. Define $\chi_k = \chi_k(\boldsymbol{y}; \{\boldsymbol{y}_j\})$ as solutions of

$$\nabla_y \cdot (B(\boldsymbol{y})\, (\nabla_y \chi_k + \boldsymbol{e}_k)) = 0. \tag{3.11b}$$

The solution χ_k gives the pressure variations in the microscopic scale in response to a unit macroscopic pressure gradient in the $\boldsymbol{e}_k$ direction. Then (3.11a) is satisfied if we let

$$p_1 = \sum_{k=1}^{3} \chi_k(\boldsymbol{y}) \frac{\partial \bar{p}(t, \boldsymbol{x})}{\partial x_k}. \tag{3.12}$$

The solution of (3.11b) is a complicated problem, and in general only qualitative properties of χ_k are known. In the periodic case (3.11b) is elementary and can be solved numerically by standard methods. The analysis of (3.11b) is a canonical problem that arises very frequently in the study of inhomogeneous media. Here we shall assume a suitable solution has been constructed and continue with the terms of next order, order ϵ^0, when (3.10) is inserted in (3.6). We obtain

$$\nabla_y \cdot (B(\boldsymbol{y})\, \nabla_y p_2) + \nabla_y \cdot (B(\boldsymbol{y})\, \nabla_x p_1) + \nabla_x \cdot (B(\boldsymbol{y})\, \nabla_y p_1) + \nabla_x \cdot (B(\boldsymbol{y})\, \nabla_x \bar{p}) - A(\boldsymbol{y})\, \bar{p}_{tt} = 0. \tag{3.13}$$

Averaging with respect to $\boldsymbol{y}$ over a period cell (or averaging over $\{\boldsymbol{y}_j\}$ in general) we obtain the following solvability condition for (3.13):

$$\nabla_x \cdot (\langle B(\nabla_y p_1 + \nabla_x \bar{p})\rangle) - \langle A\rangle\, \bar{p}_{tt} = 0. \tag{3.14}$$

If we define (q_{kl}) by

$$q_{kl} = \left\langle B\left(\frac{\partial \chi_k}{\partial y_l} + \delta_{kl}\right)\right\rangle, \quad k, l = 1, 2, 3, \tag{3.15}$$

then (3.14) coincides with (3.8). From the governing equation (3.11b) for χ_k one can easily deduce that

$$q_{kl} = \sum_{j=1}^{3} \left\langle B\left(\frac{\partial \chi_k}{\partial y_j} + \delta_{kj}\right)\left(\frac{\partial \chi_l}{\partial y_j} + \delta_{lj}\right)\right\rangle, \quad k, l = 1, 2, 3, \tag{3.16}$$

from which the symmetry and positive-definiteness of q_{kl} follow.

This completes the derivation of (3.8).

We now make several remarks on the solution found above.

(1) It is interesting to compare the pressure field p for the finite-volume-fraction

problem with that for the small-volume-fraction problem, which was described at the end of §2. For this finite-volume-fraction problem the leading-order pressure term $\bar{p}$ is locally uniform in both the liquid and the gas. However, macroscopic variations (on the scale $\boldsymbol{x}$) of $\bar{p}$ cause microscopic variations (on the scale $\boldsymbol{y}$) of p_1 as shown in (3.12).

(2) Note also that the fluid velocity is found from $\boldsymbol{u}_t = -B\nabla p$, which to leading order is $\boldsymbol{u}_t = -B(\nabla_x \bar{p} + \nabla_y p_1)$. According to (3.11 a), $\nabla_y \cdot \boldsymbol{u}_t = 0$, so that $\boldsymbol{u}$ is locally incompressible (if it is incompressible at $t = 0$). This shows that for a finite volume fraction of gas the bubble expansions and contractions are limited by the low compressibility of the liquid. Accordingly, the monopole pressure field, which was dominant for small volume fraction, is suppressed at finite volume fraction. Therefore the pressure variations in the gas for finite volume fraction are on a much smaller scale than those for small volume fraction.

(3) These results show that the limit (3.1) is not interchangeable with the low-gas-bubble-inertia limit $\tau \to 0$, in contrast with the interchangeability of the limits (2.14) and $\tau \to 0$. If we take τ to zero after the limit (3.1), the pressure gradient within a bubble is not zero.

(4) The restriction (3.2) to low frequencies is important in this problem. At higher frequencies the expansion of the pressure (3.10) will not be valid.

(5) By continuing our analysis it is possible to determine the leading-order linear correction to the bubble surface. In particular, this correction contains a mode that accounts for a drift of the bubble relative to the liquid. In the low-volume-fraction limit ($\epsilon \ll \beta \ll 1$) the drift velocity is equal to three times the background flow velocity (see e.g. Van Wijngaarden 1972).

(6) We note that, even though the fluid and gas are to leading order incompressible on the microscopic, $\boldsymbol{y}$ scale, e.g. (3.11 a, b), the mixture is compressible to leading order on the macroscopic scale. This can be seen in the effective equations.

(7) In formulating the microscopic equations (2.15)–(2.18), the small-volume-fraction equation (2.26) and the larger-volume-fraction equation (3.8) we have introduced a large number of dimensionless parameters, some of which are dependent on others. We now list the independent parameters for each problem. For the original microscopic system (2.15)–(3.18) the independent parameters are δ, ϵ, ζ, γ and τ. Note that the scaled sound speed C is given by

$$C^2 = \left[1 - \frac{4\pi}{3\epsilon^3}\left(\delta^3 - \frac{\delta}{\zeta}\right)\right]^{-1}.$$

In the limit (2.14) resulting in (2.26), the volume fraction β is negligibly small, and as a result the density ratio τ does not affect the system, so that two parameters drop out; the independent parameters are χ, ζ and γ (with χ related to ϵ, δ by (2.14) and $C^2 = (1 - \frac{4}{3}\pi\chi\zeta^{-1})^{-1}$ to order β). Similarly, in the small-frequency limit (3.1), (3.2) several of the parameters combine, and the independent parameters for (3.8) are β, τ, and $\langle A \rangle$ (defined by (3.9)).

Let us next discuss the effective equation (3.8). We shall assume that q_{ij} is isotropic, i.e. $q_{ij} = \delta_{ij} q$, since the bubble-centre distribution will be stationary and isotropic. Reverting to dimensional variables, we see that (3.8) is the wave equation

$$C_{\text{eff}}^{-2} \bar{p}_{tt} - \Delta \bar{p} = 0, \tag{3.17}$$

with the effective sound speed given by

$$C_{\text{eff}}^2 = q\left[\frac{1-\rho}{c_\ell^2} + \frac{\beta \rho_\ell}{\gamma p_0}\right]^{-1}. \tag{3.18}$$

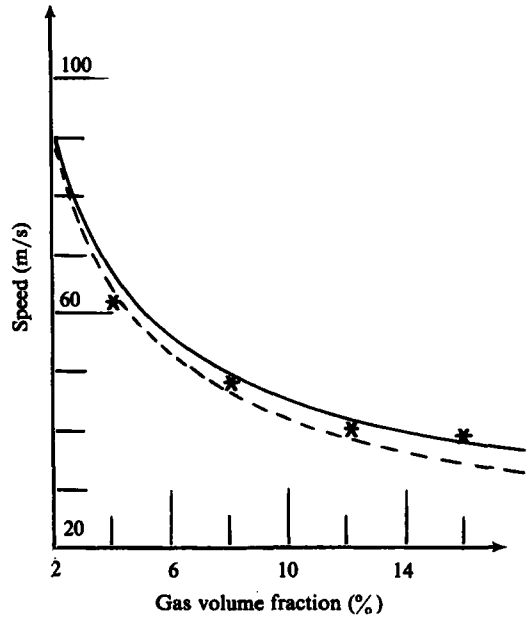

FIGURE 1. Effective velocity versus gas volume fraction at frequencies in the range 20–80 Hz and at a pressure of 1.12 bar. The broken curve is (3.18) with $q = 1$, i.e. the well-known result from (2.26). The continuous curve is (3.21). The points are from Micaelli (1982), chapter IX).

Here q is a function of the volume fraction β and the density ratio τ. When the volume fraction β is small (but still much larger than ϵ), we may use the Maxwell–Clausius–Mossotti approximation for q (Landauer 1978):

$$\frac{q-1}{q+2} = \beta\left(\frac{1}{\tau}-1\right)\Big/\left(\frac{1}{\tau}+2\right) \quad (\beta \ll 1). \tag{3.19}$$

When τ is small this gives

$$q = \frac{1+2\beta}{1-\beta}, \tag{3.20}$$

and hence for small β, we have

$$C_{\text{eff}}^2 = \frac{1+2\beta}{1-\beta}\Big/\left(\frac{1-\beta}{c_\ell^2}+\frac{\beta\rho_\ell}{\gamma p_0}\right). \tag{3.21}$$

Formula (3.18) for the effective sound speed C_{eff} is very similar to the effective sound speed $\bar{c}$ derived in I in the limit (2.14). According to (5.11) of I (with the correct $O(\beta)$ term), $\bar{c}$ is given by

$$\bar{c}^2 = \left[\frac{1-\beta}{c_\ell^2}+\frac{\beta\rho_\ell}{\gamma p_0}\right]^{-1}, \tag{3.22}$$

and so C_{eff} and $\bar{c}$ are the same except for the factor $q(\beta)$. This is remarkable, because the analyses and the modes of oscillation in the limits (3.1) and (2.14) are very different.

At small volume fraction (but finite with respect to ϵ), for example $\beta = 0.1$ (10 %), the square of the effective sound speed (3.21) is bigger than that given by (3.22) by about 13 % in this case. Comparison with the experimental results of Micaelli (1982) in figure 1 shows that (3.21) provides a better fit than (3.22) at larger β. In Micaelli's experiments the typical bubble diameter is 4 mm and the frequencies vary around

50 Hz, so viscous effects are not important. It should also be pointed out that the factor q, as given by the dipole approximation (3.19) or (3.20), is essentially the same as the one obtained by Crespo (1969). In (3.18) or (3.21) the factor q is a consequence of bubble interactions at higher volume fractions. It includes the effects of the relative compressibility of the gas–liquid mixture, and accounts for the motion of the bubble centres. This effect is also included in van Wijngaarden (1976*a*,*b*).

4. Uniform analysis in the linear low-frequency regime at finite and small gas volume fraction

In this section we shall derive from (3.6) an effective equation that contains both (3.17) and (2.26) as special cases and therefore interpolates between the Foldy approximation in the limit (2.14) and homogenization in the limit (3.1). The analysis is, however, restricted at present to periodic bubble configurations.

It is sufficient to analyse (3.6) for time-harmonic fields $p \to e^{ist} p$, so that we have the equation

$$\nabla \cdot (B \nabla p) + A s^2 p = 0, \tag{4.1}$$

with B and A defined by (3.4) and (3.5) and with $\{y_j\}$ a *periodic* configuration of period one (that is, $\{y_j\}$ are points of the three-dimensional unit lattice $\mathbb{Z}^3$). In both of the limits (2.14) and (3.1) the interbubble distance ϵ tends to zero. However, in (2.14) $\xi = \delta/\epsilon$ also goes to zero like ϵ^2 with ζ kept fixed, while in (3.1) ξ and $\zeta\delta^2$ are fixed.

To get an effective equation that contains both these limits we concentrate on the role of the parameter ϵ and think of p as a function of $\boldsymbol{x}$ and $\boldsymbol{y} = \boldsymbol{x}/\epsilon$, periodic in $\boldsymbol{y}$ for each $\boldsymbol{x}$. Then (4.1) takes on an expanded, multiscale form

$$\frac{1}{\epsilon^2}\nabla_y \cdot (B(y)\nabla_y p(x,y)) + \frac{1}{\epsilon}\nabla_y \cdot (B(y)\nabla_x p(x,y))$$
$$+\frac{1}{\epsilon}\nabla_x \cdot (B(y)\nabla_y p(x,y)) + \nabla_x \cdot (B(y)\nabla_x p) + A(y)s^2 p(x,y) = 0. \tag{4.2}$$

Let us assume that the periodic bubble configuration is centred at the origin. Then the variable $\boldsymbol{y}$ ranges over the unit period cell, which is the set of points in space with each coordinate in the interval $[-\frac{1}{2}, \frac{1}{2}]$.

Introduce a periodic function $\mu(\boldsymbol{y})$ defined by

$$\mu(y) = \begin{cases} \mu & \text{if } y \text{ is in the period cell but } |y| > \xi, \\ a & \text{if } |y| < \xi, \end{cases} \tag{4.3}$$

where μ is a constant to be defined later and

$$a = \frac{s^2\epsilon^2}{\gamma\zeta\delta^2}. \tag{4.4}$$

Let $\mathbb{1}_{\mathrm{L}}(y)$ be the indicator function of the liquid region:

$$\mathbb{1}_{\mathrm{L}}(y) = \begin{cases} 1 & \text{if } y \text{ is in the cell but } |y| > \xi, \\ 0 & \text{if } |y| < \xi. \end{cases} \tag{4.5}$$

Then we can rewrite (4.2) in the form

$$\frac{1}{\epsilon^2}[\nabla_y\cdot(B\nabla_y p)+\mu p]+\frac{1}{\epsilon}\nabla_y\cdot(B\nabla_x p)+\frac{1}{\epsilon}\nabla_x\cdot(B\nabla_y p)$$
$$+\nabla_x(B\nabla_x p)+\left(\frac{s^2}{C^2}-\frac{\mu}{\epsilon^2}\right)\mathbb{1}_{\mathrm{L}}p=0. \quad (4.6)$$

To analyse (4.6), introduce the periodic eigenvalue problem

$$\nabla_y\cdot(B(y)\nabla_y\phi(y))+\mu\phi(y)=0, \quad \phi(y) \text{ periodic}, \quad (4.7)$$

$$\langle\phi^2\rangle=\int_{\text{periodic cell}}\phi^2(y)\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3=1, \quad (4.8)$$

$$\phi(y) \text{ and } \boldsymbol{n}\cdot B(y)\nabla_y\phi(y) \text{ continuous across } |y|=\xi. \quad (4.9)$$

From the definition (4.3) of μ we see that μ is now an eigenvalue parameter. We choose in (4.7) the minimal eigenvalue μ. In that case the corresponding eigenfunction ϕ is a positive periodic function (cell function). This positivity of ϕ ensures that the cell problem (4.15) will be elliptic and hence solvable.

The term μp in the first bracket and μ/ϵ^2 in the last bracket will be treated as being of order one. Then p in (4.6) can be expanded in a power series

$$p=p_0+\epsilon p_1+\epsilon^2 p_2+\dots. \quad (4.10)$$

For p_0 we get

$$\nabla_y\cdot(B(y)\nabla_y p_0)+\mu p_0=0, \quad (4.11)$$

so from (4.7)–(4.9) we conclude that

$$p_0(x,y)=\phi(y)\,\bar{p}(x). \quad (4.12)$$

For p_1 we get

$$\nabla_y\cdot(B(y)\nabla_y p_1)+\mu p_1+\nabla_y\cdot(B(y)\nabla_x p_0)+\nabla_x\cdot(B(y)\nabla_y p_0)=0. \quad (4.13)$$

Multiplying by $\phi(y)$ and integrating over a period cell, we see that the inhomogeneous term in (4.13) satisfies the necessary solvability condition. Thus we can solve for p_1 in (4.13). In fact, let

$$p_1(x,y)=\sum_{k=1}^{3}\chi_k(y)\frac{\partial p_0}{\partial x_k}=\phi(y)\sum_{k=1}^{3}\chi_k(y)\frac{\partial\bar{p}(x)}{\partial x_k}, \quad (4.14)$$

with $\chi_k(y)$ periodic and satisfying

$$\nabla_y\cdot[B(y)\phi^2(y)\nabla_y\chi_k(y)]+\nabla_y\cdot[B(y)\phi^2(y)]\,\boldsymbol{e}_k=0. \quad (4.15)$$

Here $\boldsymbol{e}_k$, $k=1,2,3$, are unit vectors in the coordinate directions of $\mathbb{R}^3$. With this definition of χ_k and p_1, we have a solution of (4.13). As with (3.11), (4.15) cannot be solved exactly. It can, however, be analysed numerically (only in the periodic case, of course) and qualitatively.

We continue with the equation for p_2, which is

$$\nabla_y\cdot(B\nabla_y p_2)+\mu p_2+\nabla_y\cdot(B\nabla_x p_1)+\nabla_x\cdot(B\nabla_y p_1)+\nabla_x\cdot(B\nabla_x p_0)+\left(\frac{s^2}{C^2}-\frac{\mu}{\epsilon^2}\right)\mathbb{1}_{\mathrm{L}}p_0=0. \quad (4.16)$$

Multiplying (4.16) by $\phi(y)$ and integrating over a period cell, we obtain the necessary

solvability condition for the inhomogeneous term. When (4.12) and (4.14) are used, this condition becomes an equation for $\bar{p}(\boldsymbol{x})$, the effective equation

$$\sum_{i,j=1}^{3} q_{ij}^{\phi} \frac{\partial^2 \bar{p}(\boldsymbol{x})}{\partial x_i \partial x_j} + \langle \phi^2 \mathbb{1}_{\mathrm{L}} \rangle \left(\frac{s^2}{C^2} - \frac{\mu}{\epsilon^2} \right) \bar{p}(\boldsymbol{x}) = 0, \tag{4.17}$$

where, after some rearranging,

$$q_{ij}^{\phi} = \left\langle B\phi^2 \left(\frac{\partial}{\partial y_i} \chi_j + \delta_{ij} \right) \right\rangle. \tag{4.18}$$

Using both (4.15) and (4.17), we can rewrite (4.18) in the form

$$q_{ij}^{\phi} = \sum_{l=1}^{3} \left\langle B\phi^2 \left(\frac{\partial}{\partial y_l} (\chi_j) + \delta_{il} \right) \left(\frac{\partial}{\partial y_l} (\chi_j) + \delta_{jl} \right) \right\rangle, \tag{4.18'}$$

from which symmetry and positive-definiteness follow.

Let us now see how the effective equation (4.17) reduces to (2.26) and (3.17). Consider first the limit (3.1) that results in (3.17). We rewrite the eigenvalue problem (4.7)–(4.9) more explicitly:

$$(\Delta_y + \mu)\, \phi(\boldsymbol{y}) = 0 \quad \text{if } \boldsymbol{y} \text{ is in the period cell, } |\boldsymbol{y}| > \xi, \tag{4.19}$$

$$\left(\frac{1}{\tau} \Delta_y + \frac{s^2 \epsilon^2}{\gamma \zeta \delta^2} \right) \phi(\boldsymbol{y}) = 0 \quad \text{if } |\boldsymbol{y}| < \xi, \tag{4.20}$$

$$\phi(\boldsymbol{y}) \text{ and } \boldsymbol{n} \cdot B(\boldsymbol{y}) \nabla_y \phi(\boldsymbol{y}) \text{ continuous}, \tag{4.21}$$

$$\int_{\text{period cell}} \phi^2 \, \mathrm{d}\boldsymbol{y} = 1, \quad \phi(\boldsymbol{y}) \text{ periodic}. \tag{4.22}$$

For the limit (3.1) $\zeta\delta^2$ is fixed, so, since $a = s^2\epsilon^2(\zeta\gamma\delta^2)^{-1}$ in (4.4), we can carry out an expansion in powers of ϵ^2 for ϕ and μ. An elementary computation yields

$$\phi(\boldsymbol{y}) = 1 + O(\epsilon^2), \tag{4.23}$$

$$\mu = -\frac{\beta s^2}{(1-\beta)\gamma\zeta\delta^2} \epsilon^2 + \dots. \tag{4.24}$$

Similarly, from (4.19) and (4.15) we see that as $\phi \to 1$

$$q_{ij}^{\phi} \to q_{ij}, \tag{4.25}$$

where q_{ij} is given in (3.16). Therefore (4.17) becomes

$$\sum_{i,j=1}^{3} q_{ij} \frac{\partial^2 \bar{p}(\boldsymbol{x})}{\partial x_i \partial x_j} + \left(\frac{1-\beta}{C^2} + \frac{\beta}{\gamma\zeta\delta^2} \right) s^2 \bar{p}(\boldsymbol{x}) = 0, \tag{4.26}$$

which is identical with the time-harmonic version of (3.17).

Secondly, let us see how (4.17) reduces to (2.26) in the limit (2.14). Now ζ is fixed and $\xi \to 0$ so we write the constant a of (4.4) in the form

$$a = \frac{s^2\epsilon^2}{\gamma\zeta\delta^2} = \frac{s^2}{\gamma\zeta\xi^2}, \tag{4.27}$$

and (4.17)–(4.19) in the form

$$(\Delta_y + \mu)\, \phi(\boldsymbol{y}) = 0 \quad \text{if } \boldsymbol{y} \text{ is in the period cell, } |\boldsymbol{y}| > \zeta, \tag{4.28}$$

$$\left(\frac{1}{\tau} \Delta_y + \frac{s^2}{\gamma\zeta} \frac{1}{\xi^2} \right) \phi(\boldsymbol{y}) = 0 \quad \text{if } |\boldsymbol{y}| < \xi, \tag{4.29}$$

$$\int_{\text{period cell}} \phi^2(y)\,\mathrm{d}y = 1, \quad \phi(y)\text{periodic}, \tag{4.30}$$

$$\phi(y) \text{ and } \boldsymbol{n}\cdot B(y)\nabla_y \phi(y) \quad \text{continuous across } |y| = \xi. \tag{4.31}$$

The asymptotic analysis of the eigenvalue problem (4.27)–(4.31) is somewhat intricate if more than the first term is needed. The first term can be obtained easily, however, and, without spelling out the details, the results are

$$\phi(\eta) \to 1, \tag{4.32}$$

$$\mu \sim \frac{4\pi}{1-3\gamma\zeta/s^2}\xi \quad \text{as} \quad \xi\downarrow 0, \tau\downarrow 0. \tag{4.33}$$

It is here that one sees clearly that $\tau\downarrow 0$ and $\xi\downarrow 0$ are interchangeable. It is also easy to see using (4.32) that as ξ goes to zero

$$q_{ij}^{\phi} \to \delta_{ij}. \tag{4.34}$$

Thus when (4.32)–(4.34) are used in (4.17) we obtain

$$\Delta\bar{p}(x) + \left(\frac{s^2}{C^2} - \frac{4\pi\theta\chi}{1-3\gamma\zeta/s^2}\right)\bar{p} = 0, \tag{4.35}$$

which is the time-harmonic version of (2.26) (recall that $\chi = \delta\epsilon^{-3}$).

5. Concluding remarks

We summarize our results in this paper as follows.

(i) In the continuum limit at finite gas volume fraction, limit (3.1), pressure gradients inside the bubbles have to be taken into account. This occurs because the mixture behaves nearly incompressibly to leading order.

(ii) In the same limit (3.1), the effective equation for the pressure in the linearized low-frequency regime is the wave equation (3.17). The effective sound speed is given by (3.18). At small volume fraction β the effective sound speed is given by (3.21), which fits Micaelli's (1982) data better than the well-known formula $\bar{c}^2 = \gamma p/\rho_\ell \beta$ (cf. (2.15) of I), which came from (2.26) for $\beta = O(\epsilon^2)$. See figure 1.

(iii) For the idealized case of a periodic bubble configuration we have obtained the highly dispersive wave equation (4.17) as effective equation. This equation is dispersive because μ and q_{ij}^{ϕ} are complicated functions of s^2, s being the transformed time variable in (4.1). Equation (4.17) contains both the well-known equation (2.26) and (3.17), which is not dispersive, as special cases.

Even though a periodic configuration is very special, (4.17) is interesting because it interpolates between the regime of $\beta = O(\epsilon^6)$, where the Foldy approximation holds, and the finite-volume-fraction regime. In the Foldy approximation radial bubble oscillations constitute the dominant mode of bubble motion. In the limit (3.1), but with volume fraction β not too large (say less than 10 %), the principal mode of bubble motion is a dipole oscillation, which yields the effective sound speed (3.21).

One can use (4.17) to obtain corrections to (2.26), for example. This is done by carrying out the expansion (4.32), (4.33) to higher order using Hasimoto's (1958) method. For a simple cubic lattice one obtains instead of (4.35) the equation

$$\Delta\bar{p} + \left(\frac{s^2}{C^2} - \frac{4\pi\theta\chi}{1-3\gamma\zeta/s^2} + \frac{2.84(4\pi)\chi}{(1-3\gamma\zeta/s^2)^2}\left(\frac{3\beta}{4\pi}\right)^{\frac{1}{3}}\right)\bar{p} = 0. \tag{5.1}$$

Note that the correction is of order $\beta^{\frac{1}{3}}$, which is typical of interaction effects for periodic configurations. Equation (5.1) contains the correct $O(\epsilon^2)$ terms due to interactions of bubbles in a periodic system. It does not include corrections to $\bar{p}$, which may also be $O(\epsilon^2)$, coming from the multiple-scale expansion. The extension of the result (5.1) to the nonlinear setting of our previous paper (I) for periodic configurations is given by Rubinstein (1984).

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REFERENCES

BENSOUSSAN, A., LIONS, J.-L. & PAPANICOLAOU, G. C. 1978 *Asymptotic Analysis for Periodic Structures*. North Holland.

CAFLISCH, R. E., MIKSIS, M. J., PAPANICOLAOU, G. C. & TING, L. 1985 Effective equation for wave propagation in bubbly liquids. *J. Fluid Mech.* **153**, 259–273.

CARTENSEN, E. L. & FOLDY, L. L. 1947 Propagation of sound through a liquid containing bubbles. *J. Acoust. Soc. Am.* **19**, 481–501.

CRESPO, A. 1969 Theoretical study of sound and shock waves in two-phase flow. *Phys. Fluids* **12**, 2274–2282.

FOLDY, L. L. 1945 The multiple scattering of waves. *Phys. Rev.* **67**, 107–119.

HASIMOTO, H. 1959 On the periodic fundamental solutions of the Stokes equations and their application to viscous flow past a cubic array of spheres. *J. Fluid Mech.* **5**, 317–323.

KELLER, J. B. 1977 Effective behaviour of heterogeneous media. In *Statistical Mechanics and Statistical Methods in Theory and Application* (ed. U. Landman), pp. 631–644.

KELLER, J. B. 1980 Darcy's law for flow in porous media and the two-space method. In *Nonlinear Partial Differential Equations in Engineering and Applied Science* (ed. R. L. Sternberg, A. J. Katimowski & J. S. Papadkis), pp. 429–443. Dekker.

LANDAUER, R. 1978 Electrical conductivity in inhomogeneous media. In *Electrical Transport and Optical Properties of Inhomogeneous Media* (ed. J. C. Garland & D. B. Tanner). Am. Inst. Phys.

MICAELLI, J.-C. 1982 Propagation d'ondes dans les écoulements diphasiques à bulles à deux constituants. Etude théorique et expérimentale. Thèse, Université de Grenoble.

PAPANICOLAOU, G. & VARADHAN, S. R. S. 1981 Boundary value problems with rapidly oscillating coefficients. In *Random Fields* (ed. J. Fritz, L. J. Lebowitz & D. Szász), pp. 835–873. North-Holland.

RUBINSTEIN, J. 1985 Bubble interaction effects on waves in bubbly liquids. *J. Acoust. Soc. Am.* **77**, 2061–2066.

SANCHEZ-PALENCIA, E. 1980 *Non-Homogeneous Media and Vibration Theory*. Lecture Notes in Physics, vol. 127. Springer.

VAN WIJNGAARDEN, L. 1968 On equations of motion for mixtures of liquid and gas bubbles. *J. Fluid Mech.* **33**, 465–474.

VAN WIJNGAARDEN, L. 1972 One-dimensional flow of liquids containing small gas bubbles. *Ann. Rev. Fluid Mech.* **4**, 369–394.

VAN WIJNGAARDEN, L. 1976*a* Hydrodynamic interaction between bubbles in liquid. *J. Fluid Mech.* **77**, 27–44.

VAN WIJNGAARDEN, L. 1976*b* Some problems in the formulation of the equations for gas/liquid flows. In *Theoretical and Applied Mechanics* (ed. W. T. Koiter), pp. 249–260. North-Holland.