Final Exam
Numerical ODEs 269B
Winter 2008
Due 5pm Monday March 17

1. For the heat equation $u_t = bu_{xx}$ consider the following scheme

$$\frac{v_m^{n+1} - v_m^n}{k} = (1-\alpha)b\delta_x^2 v_m^{n+1} + \alpha b \delta_x^2 v_m^n. \tag{1}$$

   Find the stability region $\Lambda(\alpha)$ for this scheme.

2. (a) Show that the following approximation is really second order accurate as indicated

$$\exp\left(k(A+B+C)\right) \approx e^{(k/2)A} e^{(k/2)B} e^{kC} e^{(k/2)B} e^{(k/2)A} \tag{2}$$

   i.e., the error is $O(k^3)$, for any three matrices A, B and C.

   (b) Use this formula to derive a second order scheme for the three-dimensional heat equation $u_t = \Delta u$ that uses a one-dimensional solver at each step. Do this using the following steps:

      i. Set $A = \frac{\partial^2}{\partial x^2}$, $B = \frac{\partial^2}{\partial y^2}$, $C = \frac{\partial^2}{\partial z^2}$.

      ii. Since the PDE $u_t = bu_{xx}$ is equivalent to $u(t+k) = e^{kA}u(t)$(and similarly for B and C), interpret (2) as a series of one-dimensional heat equations on subsequent subintervals of time.

      iii. Replace each one-dimensional heat equation by a Crank-Nicolson finite difference equation.

   (c) Analyze the stability of the resulting scheme.

3. (a) Find the eigenvalues γ and (discrete) eigenfunctions $w_{\ell m}$ of the 5 point discrete Laplacian in two dimensions with periodic boundary conditions on the square $[0, 2\pi] \times [0, 2\pi]$ with $\Delta x = \Delta y = 2\pi/N$. The γ and $w_{\ell m}$ should satisfy

$$\delta_x^2 w_{\ell m} + \delta_y^2 w_{\ell m} = \gamma w_{\ell m} \tag{3}$$

(b) Find the eigenvalues λ and eigenfunctions $\phi(x)$ for the continuous Laplacian Δ in two dimensions with periodic boundary conditions on the square $[0, 2\pi] \times [0, 2\pi]$

(c) Compare the discrete and continuous eigenvalues.

4. Consider the sum

$$E[v] = \frac{1}{2} \sum_{\ell=0}^{N-1} a_{\ell+1/2} (v_{\ell+1} - v_\ell)^2 \tag{4}$$

in which $a_{\ell+1/2} = a((\ell + 1/2)h)$ is the discretization of a given positive function $a(x)$ and v_ℓ is an unknown discrete function with fixed values of v_0 and v_N.

(a) Find a finite difference equation for the discrete function v_ℓ that minimizes $E[v]$ (with fixed values of N, h, a, v_0 and v_N).

(b) Generalize this to two-dimensions with

$$E[v] = \frac{1}{2} \sum_{\ell=0}^{N-1} \sum_{m=0}^{N-1} \left(a_{\ell+1/2,m} (v_{\ell+1,m} - v_{\ell,m})^2 + b_{\ell,m+1/2} (v_{\ell,m+1} - v_{\ell,m})^2 \right) \tag{5}$$

(c) Show that the equality of mixed derivatives

$$\frac{\partial}{\partial v_{\ell_1,m_1}} \frac{\partial}{\partial v_{\ell_2,m_2}} E[v] = \frac{\partial}{\partial v_{\ell_2,m_2}} \frac{\partial}{\partial v_{\ell_1,m_1}} E[v] \tag{6}$$

implies that the two dimensional finite difference scheme corresponds to a symmetric linear system for v.

(d) More generally show that a linear system (as in the equation of a finite difference approximation of a PDE) is symmetric if and only if it can be written as the gradient of a quadratic function.