Midterm Exam
Numerical ODEs 269A
Fall 2007

1. Consider the stiff ODE

$$\begin{aligned} y'(t) &= -\lambda(y - te^t) + (t+1)e^t \\ y(0) &= 0 \end{aligned}$$

over the interval $0 \leq t \leq 1$ in which $\lambda >> 1$, for which the exact solution is $y(t) = te^t$. For step size h, please answer the following:

   (a) What is the approximation requirement on h for forward Euler?

   (b) What is the approximation requirement on h for backward Euler?

   (c) What is the stability requirement on h for forward Euler?

   (d) What is the stability requirement on h for backward Euler?

   By "approximation requirement", we mean the requirement that the truncation error is small. Your answers should have the form $O(\lambda^k)$ for some k (i.e., you need only state the dependence on λ). Please briefly justify your answers.

2. Consider the ODE

$$\begin{aligned} y'(t) &= t^k \\ y(0) &= 0. \end{aligned}$$

   Show that fourth order Runge-Kutta (RK4) provides the exact solution for $k = 0, 1, 2, 3$. You only need to check this over a single time-step h.

3. For the ODE $y' = f(y)$ consider the numerical method

$$y_n = y_{n-1} + h\left(\alpha f(y_n) + (1-\alpha) f(y_{n-1})\right)$$

   for $1/2 < \alpha < 1$. Show that the negative real axis is contained in the region of absolute stability for this method.

4. Consider the ODE

$$\begin{aligned} y'(t) &= y(t) \\ y(0) &= 1 \end{aligned}$$

for which the exact solution is $y(t) = e^t$. Define $y_n^{(h)}$ to be the solution of forward Euler with step size h for this problem.

(a) Find a formula for $y_n^{(h)}$

(b) Evaluate the terms of order h and h^2 in the global error $e_n = y(t = nh) - y_n^{(h)}$.

(c) Do the same for the Richardson extrapolation $\tilde{y}_n = 2y_{2n}^{(h/2)} - y_n^{(h)}$ and its global error $\tilde{e}_n = y(t = nh) - \tilde{y}_n$.

Hint: Use the binomial expansion

$$(1 + x)^k = 1 + kx + \frac{k(k-1)}{2}x^2 + \ldots.$$