Midterm Solution Set

Numerical ODEs 269A

Fall 2007

1. (a) Since the solution $y(t) = te^t$ is independent of λ, the approximation requirement (small truncation error) is independent of λ.

$\boxed{h = O(\lambda^0)}$

(b) $\boxed{h = O(\lambda^0)}$ for same reason

(c) For forward Euler, the stability requirement is that λh is bounded, i.e. $\boxed{h = O(\lambda^{-1})}$

(d) For backward Euler, there is no stability requirement.

$\boxed{h = O(\lambda^0)}$

2. For $n=1$, with $t_0=0$, $y_0=0$ and $f(t)=t^k$, RK4 becomes

$$Y_1 = 0$$
$$Y_2 = 0 \quad (\text{or } 1 \text{ if } k=0)$$
$$Y_3 = \frac{h}{2}\left(\frac{h}{2}\right)^k = \left(\frac{h}{2}\right)^{k+1}$$
$$Y_4 = h\left(\frac{h}{2}\right)^k = 2\left(\frac{h}{2}\right)^{k+1}$$
$$y_1 = \frac{h}{6}\left(0^k + 2\left(\frac{h}{2}\right)^k + 2\left(\frac{h}{2}\right)^k + h^k\right)$$
$$= \frac{0^k + 1 + 2^{2-k}}{6}\, h^{k+1} \qquad 0^k = \begin{cases} 1 & k=0 \\ 0 & k>0 \end{cases}$$
$$= c_k h^{k+1}$$

$$c_k = \frac{1}{6}\begin{cases} 1+1+4 \\ 1+2 \\ 1+1 \\ 1+\tfrac{1}{2} \\ 1+\tfrac{1}{4} \end{cases} = \begin{cases} 1 & k=0 \\ \tfrac{1}{2} & k=1 \\ \tfrac{1}{3} & k=2 \\ \tfrac{1}{4} & k=3 \\ 5/24 & k=4 \end{cases}$$

So $c_k = (k+1)^{-1}$ for $k = 0, 1, 2, 3$
and for these values of k

$$y_1 = \int_0^h h^k\, dk = (k+1)^{-1} h^{k+1}$$

3. Set $f = \lambda y$

Then $y_n = y_{n-1} + h(\alpha \lambda y_n + (1-\alpha)\lambda y_{n-1})$

$$(1-\alpha z) y_n = (1 + (1-\alpha) z) y_{n-1} \qquad z = \lambda h$$

$$y_n = r(z) y_{n-1}$$

with

$$r(z) = \frac{1 + (1-\alpha) z}{1 - \alpha z}$$

Note that r is an increasing function of z

and $r(0) = 1$

Also $r(z) = -1$ if $-(1-\alpha z) = 1 + (1-\alpha) z$

if $z = \frac{-2}{1-2\alpha}$

This value is negative iff $\alpha < \frac{1}{2}$

So

$|r(z)| < 1$ for all $z < 0$ if $\alpha \geq \frac{1}{2}$

$|r(z)| > 1$ for $z < -\frac{2}{1-2\alpha}$ if $\alpha < \frac{1}{2}$.

4. (a) $y_n^{(h)}$ satisfies

$$y_n^{(h)} = y_{n-1}^{(h)} + h\,y_{n-1}^{(h)} \qquad y_0^{(h)} = 1$$
$$= (1+h)\,y_{n-1}^{(h)}$$

It follows that

$$y_n^{(h)} = (1+h)^n\, y_0^{(h)}$$
$$\boxed{y_n^{(h)} = (1+h)^n}$$

(b) $y(t=nh) = e^{nh} = 1 + nh + \frac{1}{2}n^2h^2 + O(h^3)$

$$y_n^{(h)} = (1+h)^n = 1 + nh + \frac{n(n-1)}{2}h^2 + O(h^3)$$

So $e_n = \dfrac{n^2 - (n(n-1))}{2}h^2 + O(h^3)$

$$= \frac{n}{2}h^2 + O(h^3)$$

(c) $\tilde{y}_n = 2\,y_{2n}^{(h/2)} - y_n^{(h)}$ Note: Typo "$y_n^{(h/2)}$" $\to y_{2n}^{(h/2)}$

$$= 2\left(1+h/2\right)^{2n} - (1+h)^n$$
$$= 2\left\{1 + 2n\left(\frac{h}{2}\right) + \frac{2n(2n-1)}{2}\left(\frac{h}{2}\right)^2\right\}$$
$$- \left\{1 + nh + \frac{n(n-1)}{2}h^2\right\} + O(h^3)$$
$$= 1 + nh + \underbrace{\left(2\,\frac{2n(2n-1)}{2}\,\frac{1}{4} - \frac{n(n-1)}{2}\right)}_{\frac{(2n^2-n)-(n^2-n)}{2} = \frac{n^2}{2}} h^2$$
$$= 1 + nh + \frac{n^2}{2}h^2 + O(h^3)$$

$$e_n = O(h^3)$$