Midterm Exam Solution Set

Math 266A Fall 2006

1. (a) This is well-posed, since $f(u) = \cos(u)$ is smooth

   (b) This is well-posed locally, since $f(u) = \sqrt{u}$ is smooth near the initial point $u=1$

   (c) This is not well-posed. For example $u(t) = 1$ $\forall t$ is a solution. So is $u(t) = \frac{1}{12}t^3 + 1$

2. (a) $A_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ has e-values λ_k and e-vectors v_k

$$\lambda_1 = 3, \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1 \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So $$u = a_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + a_2 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(b) $$e^{tA_2} = \begin{pmatrix} e^t & 2te^t \\ 0 & e^t \end{pmatrix}$$

So
$$u = \begin{pmatrix} e^t & 2t\,e^t \\ 0 & e^t \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2 e^t \begin{pmatrix} 2t \\ 1 \end{pmatrix}.$$

3. $u_{tt} + u_t + (u^2 - b^2) = 0$

WLOG take $b > 0$.

Stationary points are $U_1 = \begin{pmatrix} u_1 \\ u_{t1} \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$ and $U_2 = \begin{pmatrix} u_2 \\ u_{t2} \end{pmatrix} = \begin{pmatrix} -b \\ 0 \end{pmatrix}$

The linearized eqtns are

$$\begin{pmatrix} u' \\ u_t' \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ -2u & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \mp 2b & -1 \end{pmatrix}$$

with e-values

$u_1 = b$: $\quad \lambda(\lambda+1) + 2b = 0 \Rightarrow \lambda = \frac{1}{2}\left(-1 \pm \sqrt{1-8b}\right)$

$u_2 = -b$: $\quad \lambda(\lambda+1) - 2b = 0 \Rightarrow \lambda = \frac{1}{2}\left(-1 \pm \sqrt{1+8b}\right)$

For $b > \frac{1}{8}$, $\quad u_1 = b$: λ has complex e-value with neg. real part

$\Rightarrow$ a stable spiral

$u_2 = -b$: λ has pos and neg values

$\Rightarrow$ a saddle

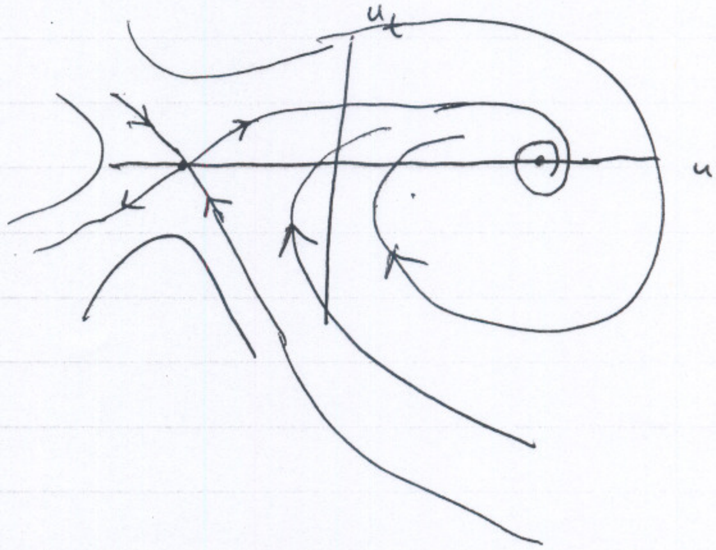
For $b < \frac{1}{8}$, $\quad u_1 = b$: λ has neg values

$\Rightarrow$ a stable node

$u_2 = -b$: λ has pos and neg values

$\Rightarrow$ a saddle

For $b > \frac{1}{8}$



4. $u_{tt} = u(1-u^2) = F'(u)$

$$F(u) = u^2\left(\tfrac{1}{2} - \tfrac{1}{4}u^4\right)$$

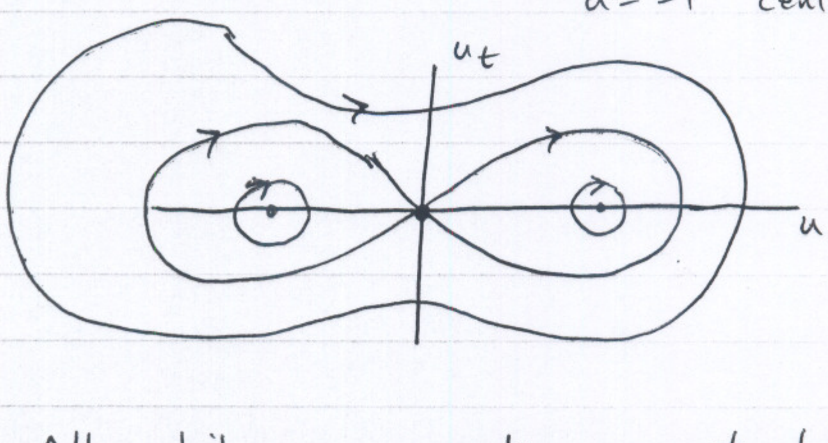
$$\cancel{\tfrac{1}{2}u_t^2 = u^2\left(\tfrac{1}{2} - \tfrac{1}{4}u^4\right) + \text{constant}}$$

$$E = \tfrac{1}{2}u^2 - u^2\left(\tfrac{1}{2} - \tfrac{1}{4}u^4\right)$$

Orbits are level curves for E

Stationary points are $u=0$ saddle

$u = \pm 1$ centers



All orbits are periodic, except stationary points and homoclinic orbits starting (and ending) at saddle $u=0$.