Math 266A: Final Exam

December 9, 1998

1. (i) Consider an autonomous system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ in which $\mathbf{y}$ is an n-vector. Suppose that $\mathbf{z}(t)$ is a periodic solution of this system with period T, and that every nearby solution is also periodic. Show that the corresponding Floquet matrix $\Omega = \mathbf{F}_\xi(\mathbf{z}(0), T)$ for the linearized system (in which $\mathbf{F}(\xi, t)$ is the general solution with initial data ξ) has $\omega = 1$ as an eigenvalue with multiplicity greater than 1.

(ii) Show that such a system is not structurally stable; i.e. there are small perturbations of f for which there is not periodic solution near z.

2. Calculate the Green's function for the following differential operators:

(i) $Lu = u_{xx} - 2x^{-1}u_x + 2x^{-2}u$ for $1 < x < 2$ with $u(1) = u(2) = 0$.

(ii) $Lu = u_{xx} + u$ for $0 < x < 1$ with $u(0) = u(1)$ and $u_x(0) = u_x(1)$.

3. (i) Consider a periodic solution $z(t)$ for a second order autonomous system $u_{tt} = f(u)$, in which $f(u) = F_u(u)$. For simplicity assume that $f(0) = 0$ and that $(u, u_t) = (0, 0)$ is the only stationary point inside the orbit of z. Show that the period T of z is given by

$$T = 2\int_{z_1}^{z_2} \frac{du}{\sqrt{2F(u)}}$$

in which $z_1 < 0$ and $z_2 > 0$ are the values of $z(t)$ at which $z'(t) = 0$.

(ii) For the nonlinear pendulum equation

$$\begin{aligned} u_{tt} &= -\sin(u) \\ u(0) &= u_0 \\ u_t(0) &= 0 \end{aligned}$$

show that the period is an increasing function of the "amplitude" $|u_0|$. For simplicity, consider only the periodic orbits centered around 0.

4. (i) Let $Lu = u_{xx} - a(x)^2 u$ on $0 < x < 1$ with boundary conditions $u(0) = u(1) = 0$, in which a is not identically 0. Show that the eigenvalues λ_i of L are all negative.

(ii) Show that the largest eigenvalue λ_0 is given by

$$\lambda_0 = -\inf_u |(Lu, u)|/(u, u)$$

in which the inf is taken over smooth functions satisfying the boundary conditions. The inner product is $(f, g) = \int_0^1 f(x)g(x)dx$.

(iii) Consider the same L but with boundary conditions $u_x(0) = u_x(1) = 0$. Show that the largest eigenvalue κ_0 satisfies the same variational formula, but without the restriction that u satisfies the boundary conditions.

(iv) Show that $\lambda_0 < \kappa_0$.