Solution Set
Final Math 266A
Fall 2006

1. (a)

$$u = -2\alpha\left((x-x_1)^{-1} + (x-x_2)^{-1}\right)$$
$$u_t = -2\alpha\left(\dot{x}_1 (x-x_1)^{-2} + \dot{x}_2 (x-x_2)^{-2}\right)$$
$$u_x = 2\alpha\left((x-x_1)^{-2} + (x-x_2)^{-2}\right)$$
$$u_{xx} = -4\alpha\left((x-x_1)^{-3} + (x-x_2)^{-3}\right)$$

$$u u_x = -4\alpha^2\left((x-x_1)^{-3} + (x-x_2)^{-3} + (x-x_1)^{-1}(x-x_2)^{-2} + (x-x_1)^{-2}(x-x_2)^{-1}\right)$$

Now

$$(x-x_1)^{-1}(x-x_2)^{-2} = \frac{a}{(x-x_1)} + \frac{b}{(x-x_2)} + \frac{c}{(x-x_2)^2}$$
$$= (x-x_1)^{-1}(x-x_2)^{-2}\left\{a(x-x_2)^2 + b(x-x_1)(x-x_2) + c(x-x_1)\right.$$

$$\Rightarrow \quad 1 = a(x-x_2)^2 + b(x-x_1)(x-x_2) + c(x-x_1)$$
$$= (a+b)x^2 + (-2ax_2 - b(x_1+x_2) + c)x + (ax_2^2 + bx_1x_2 - cx_1)$$
$$\Rightarrow \quad a+b = 0 \Rightarrow b = -a$$
$$c = 2ax_2 + b(x_1+x_2) = 2ax_2 - a(x_1+x_2) = a(x_2-x_1)$$
$$ax_2^2 + bx_1x_2 - cx_1 = 1$$
$$\Rightarrow 1 = ax_2^2 - ax_1x_2 - a(x_2-x_1)x_1 = ax_2(x_2-x_1) - a(x_2-x_1)x_1$$
$$= a(x_2-x_1)^2$$
$$\Rightarrow \quad a = x_{12}^{-2} \qquad b = -x_{12}^{-2} \qquad c = -x_{12}^{-1}$$
$$x_{12} = (x_1 - x_2)$$

Then

$$u u_x = -4\alpha^2\left\{(x-x_1)^{-3} + (x-x_2)^{-3} + \left(x_{12}^{-2}(x-x_1)^{-1} - x_{12}^{-2}(x-x_2)^{-1} - x_{12}^{-1}(x-x_2)^{-2}\right)\right.$$
$$\left. + \left(x_{21}^{-2}(x-x_2)^{-1} - x_{21}^{-2}(x-x_1)^{-1} - x_{21}^{-1}(x-x_1)^{-2}\right)\right.$$

Since $x_{21} = -x_{12}$

$$u u_x = -4\alpha^2 \left\{ (x-x_1)^{-3} + (x-x_2)^{-3} + x_{12}^{-1}(x-x_1)^{-2} - x_{12}^{-1}(x-x_2)^{-2} \right\}$$

Then

$$\begin{aligned} u_t + u u_x - \alpha u_{xx} &= -2\alpha \left\{ \dot{x}_1 (x-x_1)^{-2} + \dot{x}_2 (x-x_2)^{-2} \right\} \\ &\quad - 4\alpha^2 \left\{ (x-x_1)^{-3} + (x-x_2)^{-3} + x_{12}^{-1}(x-x_1)^{-2} - x_{12}^{-1}(x-x_2)^{-3} \right\} \\ &\quad + 4\alpha^2 \left\{ (x-x_1)^{-3} + (x-x_2)^{-3} \right\} \\ &= (x-x_1)^{-2}\left(-2\alpha \dot{x}_1 - 4\alpha^2 x_{12}^{-1}\right) \\ &\quad + (x-x_2)^{-2}\left(-2\alpha \dot{x}_2 + 4\alpha^2 x_{12}^{-1}\right) \\ &= 0 \end{aligned}$$

if

$$\boxed{\begin{aligned} \dot{x}_1 &= -2\alpha\, x_{12}^{-1} = -2\alpha (x_1 - x_2)^{-1} \\ \dot{x}_2 &= 2\alpha\, x_{12}^{-1} = -2\alpha (x_2 - x_1)^{-1} \end{aligned}}$$

(b) A similar calculation shows that

$$\dot{x}_k = -2\alpha \sum_{l \neq k} (x_k - x_l)^{-1}$$

2. (a) $Lu = 0$ has solution $e^{\lambda x}$ with $\lambda^2 - 2\lambda + 1 = 0$. Since this has a double root $\lambda = 1$, the second solution is xe^x.

$$u = ae^x + bxe^x$$

~~Now the Green's function is~~

The solutions $\varphi_1(x) = e^x(1-x)$ $\qquad \varphi_{1x} = (1-x-1)e^x = -xe^x$

$\qquad\qquad\qquad \varphi_2(x) = xe^x$ $\qquad \varphi_{2x} = (1+x)e^x$

has

$$\begin{pmatrix} \varphi_1(0) & \varphi_{1x}(0) \\ \varphi_2(0) & \varphi_{2x}(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$W = \begin{vmatrix} \varphi_1(x) & \varphi_{1x}(x) \\ \varphi_2(x) & \varphi_{2x}(x) \end{vmatrix} = \begin{vmatrix} e^x(1-x) & -xe^x \\ xe^x & (1+x)e^x \end{vmatrix}$$

$$= (1 - x^2 + x^2)e^{2x} = e^{2x}$$

$$K(x,y) = \begin{cases} e^{-2y}\left(e^y(1-y)\,xe^x - ye^y\,e^x(1-x)\right) & \cancel{\text{for}}\ y < x \\ 0 & \cancel{\text{for}}\ x < y \end{cases}$$

$$G(x,y) = K(x,y) + \alpha(y)\varphi_1(x) + \beta(y)\varphi_2(x)$$

with

$$\alpha(y) = K(0,y) = 0$$

$$\beta(y) = -\left(K(1,y) + \varphi_1(1)\alpha(1)\right)/\varphi_2(1)$$

$$= \cancel{-\varphi_1(1)\alpha(1)/\varphi_2(1)}$$

$$= -e^{-2y}\left(e^y(1-y)e^1 + 0\right)/e^1 = -e^{-y}(1-y)$$

$$G(x,y) = \begin{cases} -ye^{-y}e^x(1-x) & y < x \\ -e^{-y}(1-y)\,xe^x & x < y \end{cases}$$

<u>Check</u> $G_{xx} - 2G_x + 1 = 0$ for $x \neq y$ since G is a linear combination of $(1-x)e^x$ and xe^x. ✓

$$G(x=y+,y) = G(x=y-,y) = -x(1-x) \checkmark$$

$$G(0,y) = G(1,y) = 0 \checkmark$$

$$G_x(x=y+,y) = \frac{\partial}{\partial x}\left(-y\, e^{-y} e^x (1-x)\right)\Big|_y = -ye^{-y}\left(e^x(1-x-1)\right)$$

$$= y^2.$$

$$G_x(x=y-,y) = \frac{\partial}{\partial x}\left(-e^{-y}(1-y)\, x e^x\right)\Big|_y = -(1-y)e^{-y}\left(e^x(1+x)\right)$$

$$= -(1-y^2)$$

$$[G_x] = y^2 - \left(-(1-y^2)\right) = 1 \checkmark$$

(b) $Lu = \dfrac{d^2u}{dx^2} - 2\dfrac{du}{dx} + u = e^{2x}\left(e^{-2x}u_x\right)_x + u$

Set $q = e^{-2x}$

Then

$$\int_0^1 u(Lv)\, e^{-2x} dx = \int_0^1 u\, e^{2x}\left(e^{-2x} v_x\right)_x e^{-2x} dx$$

$$= \int_0^1 u\left(e^{-2x} v_x\right)_x dx$$

$$= \int_0^1 -u_x e^{-2x} v_x \, dx$$

$$= \int_0^1 \left(u_x e^{-2x}\right)_x v\, dx$$

$$= \int_0^1 e^{-2x}(Lu)\, v\, dx \checkmark$$

3. (a) $\int v L u \, dx = \int v(u_{xx} - a e^{-x^2} u) \, dx$
$$= \int -v_x u_x - a e^{-x^2} uv \quad dx$$
$$= \int v_{xx} u - a e^{-x^2} uv \quad dx$$
$$= \int (Lv) u \, dx$$

(b) $L\varphi_k = \lambda_k \varphi_k \Rightarrow \lambda_k \int \varphi_k^2 dx = \int \varphi_k (\varphi_{kxx} - a e^{-x^2} \varphi_k) dx$
$$= \int -\varphi_{kx}^2 - a e^{-x^2} \varphi_k^2 \, dx$$
$$< 0$$

so $\lambda_k < 0$

(c) $\lambda_1(a) = \inf\limits_{\|u\|=1} (u, Lu) = \inf\limits_{\|u\|=1} \int u L u \, dx$

Now $\int u L u dx = \int -u_x^2 - a e^{-x^2} u^2 dx$
This is a decreasing function of a, i.e. if $a_1 < a_2$

$$\int u L_2 u \, dx < \int u L_1 u \, dx \qquad \text{for } L_k = L(a_k)$$

Thus
$$\inf_{\|u\|=1} \int u L_2 u \, dx < \inf_{\|u\|=1} \int u L_1 u \, dx$$

$$\lambda_1(a_2) < \lambda_1(a_1)$$

4. Fixed points are $\dot\theta = 0$, $\sin\theta = 0$, i.e. $\bar\theta = 0, \pi$ (plus multiples of 2π. The type is determined by the linearized eqtn

$$\begin{pmatrix} \theta' \\ \dot\theta' \end{pmatrix}^{\cdot} = \begin{pmatrix} 0 & 1 \\ -\cos\bar\theta & -b \end{pmatrix}$$

The e-values are

$$\lambda(\lambda+b) + \cos\bar\theta = 0$$
$$\lambda^2 + b\lambda + \cos\bar\theta = 0$$
$$\lambda = \lambda_{\pm} = \tfrac{1}{2}\left(-b \pm \sqrt{b^2 - 4\cos\bar\theta}\right)$$

For $\bar\theta = \pi$, $\cos\bar\theta = -1$, then $\sqrt{b^2 - 4\cos\bar\theta} = \sqrt{b^2+4} > b$ so that $\lambda_+ > 0 > \lambda_-$. This is a saddle for all b

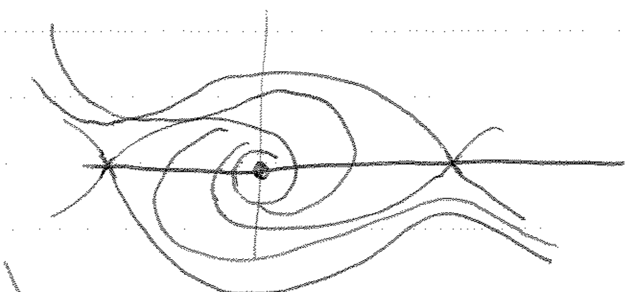
For $\bar\theta = 0$, $\cos\theta = 1$, then $\sqrt{b^2 - 4\cos\theta} = \sqrt{b^2-4}$

$b < 2$, $\sqrt{b^2-4} = i\sqrt{4-b^2}$

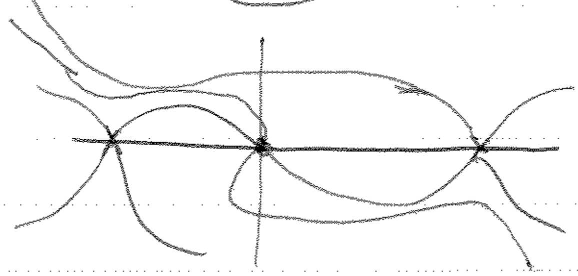
$\mathrm{Re}\,\lambda = -\frac{1}{2}b$, $\mathrm{Im}\,\lambda = \pm\sqrt{4-b^2}$

This is a stable spiral

$b > 2$, $0 < \sqrt{b^2-4} < b$, $\lambda < 0$. This is a stable node



$b < 2$



$b > 2$

5. (a) Fixed point

$$y = x = b^{-1}\frac{x}{1+x} \qquad x=0 \text{ or } x = b^{-1}-1$$

$1 = b^{-1}(1+x)^{-1}$

Two solution

(i) $(x_1, y_1) = (0,0)$

(ii) $(x_2, y_2) = (b^{-1}-1, b^{-1}-1)$

Bifurcation at $b=1$ where these two are equal

$$\frac{d}{dx}\left(\frac{x}{1+x}\right) = \frac{d}{dx}\left(1-\frac{1}{1+x}\right) = (1+x)^{-2}$$

(i)
$$\begin{pmatrix} x \\ y \end{pmatrix}^{\circ} = \begin{pmatrix} -1 & 1 \\ 1 & -b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

e-values $\lambda_\pm$: $(\lambda+1)(\lambda+b)-1 = 0$

$$\lambda^2 + (1+b)\lambda + (b-1) = 0$$

$$\lambda_\pm = \tfrac{1}{2}\left(-(1+b) \pm \sqrt{(1+b)^2 - 4(b-1)}\right)$$

For $0<b<1$, $\lambda_- < 0 < \lambda_+$

$b>1$ $\lambda_- < \lambda_+ < 0$

(ii)
$$\begin{pmatrix} x' \\ y' \end{pmatrix}^{\circ} = \begin{pmatrix} -1 & 1 \\ (1+x)^{-2} & -b \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ b^2 & -b \end{pmatrix}$$

e-values $\lambda_\pm$ = $\lambda^2 + (1+b)\lambda + (b-b^2) = 0$

$$\lambda_\pm = \tfrac{1}{2}\left(-(1+b) \pm \sqrt{(1+b)^2 - 4(b-b^2)}\right)$$

For $0<b<1$, $\lambda_- < \lambda_+ < 0$

$b>1$, $\lambda_- < 0 < \lambda_+$

Transcritical bifurcation.

$$\dot{x} = y + \mu x$$

(b) <u>Critical points</u> $\quad \dot{y} = -x + \mu y - x^2 y$

$y = -\mu x$ $\quad\Rightarrow\quad$ $x = 0$ or

$0 = -x - \mu^2 x + \mu x^3$ $\quad$ $(1-\mu^2) + \mu x^2 = 0$

$$x = -\frac{1-\mu^2}{\mu}$$

$x = y = 0$ is a fixed pt

No other fixed points near $x = y = 0$ for $\mu \approx 0$.

Polar coordinates $\quad (x, y) = r(\cos\theta, \sin\theta)$

With some work, one finds

$$\frac{d}{dt}\left(\tfrac{1}{2}r^2\right) = 2\mu r^2 - r^4 \cos^2\theta \sin^2\theta$$

$$\frac{d}{dt}\theta = -1 - r^2 \cos^2\theta \sin^2\theta$$

For $\mu < 0$, r^2 is strictly decreasing $\Rightarrow$ $\boxed{(0,0) \text{ is a stable } \cancel{\text{node}} \text{ spiral}}$

For $\mu > 0$, r^2 dominates $r^4 \cos^2\theta \sin^2\theta$ for $r \ll 1 \Rightarrow r^2$ increasing near $r = 0$

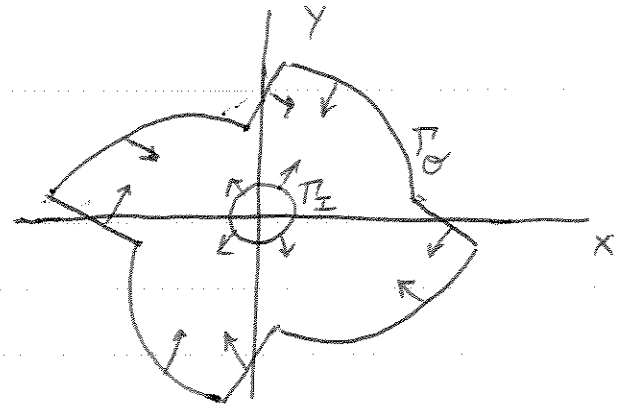
$\Rightarrow$ $\boxed{(0,0) \text{ is an unstable spiral}}$

<u>Periodic orbits</u>

For $r^2 \gg 2\mu$, r^2 is decreasing except for a small interval of θ near $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

Also ~~...~~ θ is strictly decreasing

One can thus see that flow is incoming on the following body



For $0<\mu<<1$.

On Γ_I, flow is incoming, since $(0,0)$ is an unstable node.

On Γ_O, flow is incoming on straight segments since $\dot{\theta}<0$.
flow is incoming on nearly circular arcs, because they avoid intervals around 0, $\pi/2$, π, $3\pi/2$.

For $\mu<<1$, the outer boundary Γ_O can be very small. So there is a periodic orbit by Poincare-Bendixson and it goes to $(0,0)$ as $\mu \to 0+$.