

Math 266A: Final Exam

December 6, 2006

Due: Friday December 15, 2006

Allowed time: 6 hours.

1. The viscous Burger's equation on $-\infty < x < \infty$ is

$$u_t + uu_x = \alpha u_{xx}. \quad (1)$$

- (a) Find ODEs for $x_1(t)$ and $x_2(t)$, so that (1) has a solution of the form

$$u(x, t) = -2\alpha \left((x - x_1(t))^{-1} + (x - x_2(t))^{-1} \right). \quad (2)$$

Hint: Use partial sums expansions of the following form (in which you must find b , c and d)

$$(x - x_1)^{-1}(x - x_2)^{-2} = b(x - x_1)^{-1} + c(x - x_2)^{-1} + d(x - x_2)^{-2}. \quad (3)$$

- (b) (BONUS PROBLEM) Generalize the solution (2) to a sum (which could be infinite or finite) of the form

$$u(x, t) = \sum_k a_k(t)(x - x_k(t))^{-1}. \quad (4)$$

2. Consider the operator

$$Lu = u_{xx} - 2u_x + u \quad (5)$$

on $0 \leq x \leq 1$ with boundary conditions

$$u(0) = u(1) = 0. \quad (6)$$

- (a) Find the Green's function for L . Hint: Find explicit solutions of the initial value problem for $Lu = 0$ starting at $x = 0$.
- (b) Find a function $q(x)$, so that L is self-adjoint in the inner product

$$(u, v) = \int_0^1 u(x)v(x)q(x)dx. \quad (7)$$

3. Consider the operator

$$Lu = u_{xx} - ae^{-x^2}u \quad (8)$$

on $0 \leq x \leq 1$ with boundary conditions

$$u(0) = u(1) = 0 \quad (9)$$

in which a is a positive constant.

- (a) Show that L is self-adjoint in the usual inner product.
- (b) Show that the eigenvalues λ_k ($k = 1, 2, \dots$) of L are all negative.
- (c) Show that the leading eigenvalue λ_1 is monotone decreasing with respect to a ; i.e., if $0 < a_1 < a_2$, then

$$\lambda_1(a_2) < \lambda_1(a_1) < 0. \quad (10)$$

4. Find and classify the fixed points for the damped pendulum equation

$$\ddot{\theta} + b\dot{\theta} + \sin \theta = 0 \quad (11)$$

for $b > 0$. Plot the phase portraits for the qualitatively different cases $b < 2$ and $b > 2$.

5. Bifurcations.

- (a) Find and classify the bifurcation for the following system:

$$\begin{aligned} \dot{x} &= y - x \\ \dot{y} &= -by + x/(1+x). \end{aligned} \quad (12)$$

- (b) (BONUS PROBLEM) Show that the following system has a supercritical Hopf bifurcation at $\mu = 0$ and $(x, y) = (0, 0)$:

$$\begin{aligned} \dot{x} &= y + \mu x \\ \dot{y} &= -x + \mu y - x^2 y. \end{aligned} \quad (13)$$

Hint: Transform to polar coordinates (r, θ) . Show that the critical point $r = 0$ changes from a stable spiral for $\mu < 0$ to an unstable spiral for $\mu > 0$. Use the Poincaré-Bendixson Theorem to show that there is a periodic orbit close to $r = 0$ for small μ .