Conserved Quantities

A Conserved quantity helps to simplify the phase plane analysis away from stationary points. Here are some examples.

Consider the ~~[illegible] oscillator~~ mechanics equation

$$u_{tt} = f(u) = \mathcal{F}'(u) \qquad (3.19)$$

Multiply by u_t to get

$$u_t u_{tt} = \mathcal{F}'(u)\, u_t$$

Now

$$u_t u_{tt} = \left(\tfrac{1}{2} u_t^2\right)_t$$

$$\mathcal{F}'(u)\, u_t = \frac{d}{dt} \mathcal{F}(u(t))$$

So that

$$\frac{d}{dt}\left(\tfrac{1}{2} u_t^2 - \mathcal{F}(u)\right) = 0 \qquad (3.20)$$

Integrate in time to get

~~$E(t)=$~~

$$E = \tfrac{1}{2} u_t^2 - \mathcal{F}(u) = \text{constant} \qquad (3.21)$$

This shows that $E = \frac{1}{2} u_t^2 - \mathcal{F}(u)$ is a constant c along an orbit, i.e. the orbits are the level sets of E.

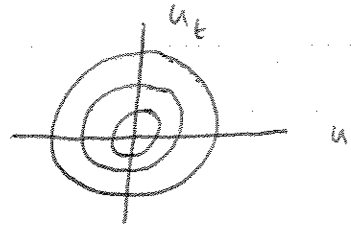
The first example is a harmonic oscillator

$$u_{tt} = -u \qquad (3.22)$$

so that

$$\begin{aligned} \mathcal{F}(u) &= -\tfrac{1}{2} u^2 \\ E &= \tfrac{1}{2} u_t^2 + \tfrac{1}{2} u^2 \end{aligned} \qquad (3.23)$$

The orbits in phase space $\{(u, u_t)\}$ are circles $u^2 + u_t^2 = 2E$ in which E is a constant.



The stationary point $(u, u_t) = 0$ is a center.

Next consider the pendulum equation

$$u_{tt} = -\sin u \tag{3.24}$$

Then

$$\begin{aligned} F(u) &= \cos u \\ E(u) &= \tfrac{1}{2} u_t^2 - \cos u \end{aligned} \tag{3.25}$$

There are two sets of stationary points:

$$\begin{aligned} S_1 &= \{ (u, u_t) = (2n\pi, 0) \} \\ S_2 &= \{ (u, u_t) = \{ ((2n+1)\pi, 0) \} \end{aligned} \tag{3.26}$$

All of the S_1 points are identical to $(0,0)$ and all of the S_2 points are identical to $(\pi, 0)$.

~~consider~~ The ODE is

$$\begin{pmatrix} u \\ u_t \end{pmatrix}_t = \begin{pmatrix} u_t \\ -\sin u \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} v \\ -\sin u \end{pmatrix} \tag{3.27}$$

using $v = u_t$.

Around a stationary point, $(u, u_t) = (u_*, v_*) = (u_*, 0)$, the linearized ~~sys~~ system is

$$\begin{pmatrix} u' \\ v' \end{pmatrix}_t = M \begin{pmatrix} u' \\ v' \end{pmatrix}$$

in which
$$M = \frac{\partial}{\partial(u,v)} \begin{pmatrix} v \\ -\sin u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\cos u & 0 \end{pmatrix} \qquad (3.28)$$

For $u = u_* = 0$, $-\cos u = -1$ and

$$M_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Since
$$\det(M - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix}$$

$$= \lambda^2 + 1$$

with roots
$$\lambda = \pm i \qquad (3.29)$$
then ~~a~~ $(0,0)$ is a center.

On the other hand, for $u = \pi$, $-\cos u = 1$
and
$$M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\det(M-\lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$$

$$= \lambda^2 - 1$$

with

$$\lambda = \pm 1 \qquad (3.30)$$

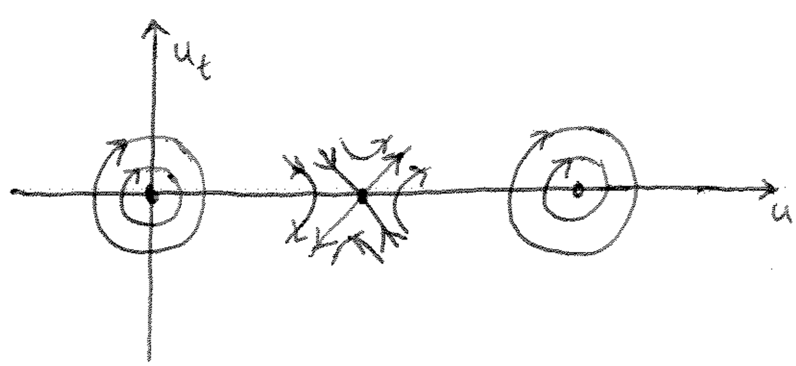
Thus $(\pi, 0)$ is a saddle. The corresponding eigenvector are

$$\varphi = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{for} \quad \lambda = -1 \text{, the stable direction} \qquad (3.31)$$

$$\psi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{for} \quad \mu = 1 \text{, the unstable direction}$$

The same conclusion can be found by showing that $(0,0)$ is a minimum for E, while $(\pi, 0)$ is a saddle point.

This shows that near the stationary points, the flow looks as shown



Analysis of E at the singular points is based on its derivatives. First $\nabla E = (\partial_u, \partial_{u_t}) E = 0$ at the stationary points. In fact

$$\frac{\partial}{\partial u} E = \sin u = 0 \quad \text{for} \quad u = n\pi$$

$$\frac{\partial}{\partial u_t} E = u_t = 0 \quad \text{for} \quad u_t = 0. \tag{3.31}$$

This is ~~also~~ always true, since ∇E is the right hand side of the ODE.

~~Other~~ Next look at the second derivative

$$\nabla\nabla E = \begin{pmatrix} E_{uu} & E_{uu_t} \\ E_{uu_t} & E_{u_t u_t} \end{pmatrix} = \begin{pmatrix} \cos u & 0 \\ 0 & 1 \end{pmatrix} \tag{3.32}$$

$$\nabla\nabla E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for } (u, u_t) = (0, 0) \tag{3.33}$$

$$\nabla\nabla E = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for } (u, u_t) = (\pi, 0) \tag{3.34}$$

Near ~~the~~ a stationary point with $E = E_0$ at the stationary point E is approximated by the ~~Taylor series~~ quadratic term

$$E \cong E_0 + \tfrac{1}{2}\left(E_{uu} u'^2 + 2 E_{uu_t} u' u_t' + E_{u_t u_t} u_t'^2\right)$$

$$= E_0 + \begin{pmatrix} u' \\ u_t' \end{pmatrix} \cdot \nabla\nabla E \begin{pmatrix} u' \\ u_t' \end{pmatrix} \tag{3.35}$$

A matrix $\overbrace{A}^{\text{self-adjoint}}$ is <u>positive definite</u> ~~iff~~ (denoted $A>0$) if

$$x \cdot Ax > 0 \tag{3.36}$$

for all $x \neq 0$. Since A is self-adjoint, it has a full set of eigenvectors v_k and eigenvalues λ_k. Moreover, every vector x can be written as

$$x = \sum_{k=1}^{n} a_k v_k \tag{3.37}$$

It follows that

$$Ax = \sum_{k=1}^{n} a_k A v_k = \sum_{k=1}^{n} \lambda_k a_k v_k \tag{3.38}$$

$$x \cdot Ax = \left(\sum_{j=1}^{n} a_j v_j \right) \cdot \sum_{k=1}^{n} \lambda_k a_k v_k$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} a_j a_k \lambda_k \, v_j \cdot v_k$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \lambda_k \, a_j a_k \, \delta_{jk}$$

$$= \sum_{k=1}^{n} \lambda_k a_k^2 \tag{3.39}$$

in which $\delta_{jk} = 0$ or 1 if $j \neq k$ or $j = k$. From this we see that A is positive definite if and only if

$$\lambda_k > 0 \quad \forall k. \tag{3.40}$$

Additional terminology: A is <u>negative definite</u> ~~iff~~ (denoted $A<0$) if

$$x\cdot Ax<0 \qquad (3.41)$$

for all $x\neq 0$, which is the case if

$$\lambda_k<0 \quad \forall k. \qquad (3.42)$$

A is ~~p~~ positive semi-definite ~~iff~~ (denoted $A\geq 0$) if

$$x\cdot Ax\geq 0 \qquad (3.43)$$

for all $x\neq 0$, which is the case if

$$\lambda_k\geq 0 \quad \forall k. \qquad (3.44)$$

Now we return to consideration of $\nabla\nabla E$. It is a self-adjoint matrix, by equality of mixed partial derivatives. At $(0,0)$, as in (3.33) its eigenvalues are $\lambda=1$ so that

$$\nabla\nabla E>0 \qquad (3.45)$$

~~As that~~ The expansion (3.35) shows that

$$E>E_0$$

if (u', u_t') near but not equal to $(0,0)$. Thus $(0,0)$ is a minimum of E and the level curves of E are closed curves, as in the figure on page 3.15.

(and $(\pi,0)$ is a saddle point)

At $(\pi, 0)$, as in (3.34), the eigenvalues are $\lambda = \pm 1$. In this case $\nabla\nabla E$ is indefinite. In some directions E is increasing as u moves from $(\pi, 0)$; in other directions it is decreasing. So there are directions in which E stays constant $E = E_0$. These are the orbits that asymptotically approach the singular point as in the picture on page 3.15