3. Autonomous Equations and Phase Plane Methods

Equations of the type

$$u_t = f(u) \qquad (3.1)$$

with f independent of t are called autonomous. The equation is called nth order, if u is an n-vector.

Any system

$$u_t = f(u,t)$$

can be made autonomous by increasing its order. Set

$$v = \begin{pmatrix} u \\ t \end{pmatrix} \qquad g(v) = g(u,t) = \begin{pmatrix} f(u,t) \\ 1 \end{pmatrix}$$

Then

$$v' = g(v)$$

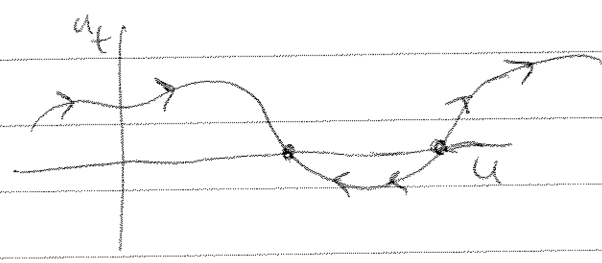
So any benefit from being autonomous must be connected to low order.

First Order Eqtns

A first order autonomous equation has the form

$$u_t = f(u) \qquad (3.2)$$

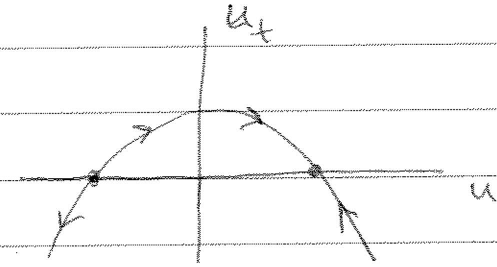
with u a scalar. Define the phase plane as the plane of value (u, u_t). Then (3.2) describes a curve in phase plane



For $u_t > 0$, u is increasing in time and for $u_t < 0$, u is decreasing. This gives a direction to a flow along this curve. The points $u_t = f(u) = 0$ are stationary points for the flow.

Example

$$u_t = (1-u^2) = (1-u)(1+u) \tag{3.3}$$



Phase plane portrait

The point $u=-1$ is unstable. The point $u=1$ is stable

The exact solution is

$$u(t) = \tanh(t+t_0) \tag{3.4}$$

Note that the solution is only determined up to a shift in time. Specifying that shift t_0 is equivalent to specifying initial data.

Second Order Eqtns

For second order equations, many of the same things still work. We will discuss the following

Methods:

- phase plane
- conserved quantities
- symmetries
- geometric arguments

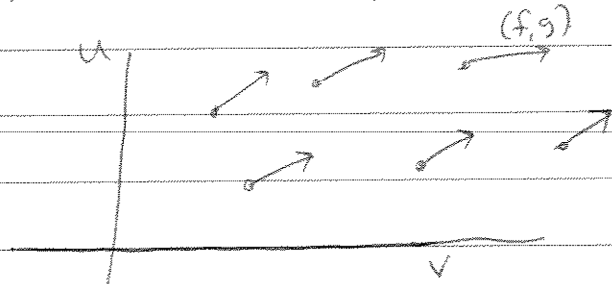
Solution Features

- stationary points
- stability
- periodic solutions
- homoclinic and heteroclinic orbits.

For a system of the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix} \qquad (3.4)$$

The phase plane the is (u,v) plane. At each point (u,v), the vector (f,g) specifies the direction (and speed) of the change in (u,v) as a function of time



Vector field (f,g) in phase plane (u,v)

An orbit is a curve $(u(t), v(t))$ whose components solve the differential equation (3.4).

A stationary point $(u,v) = (u_*, v_*)$ is a point at which $(f,g) = 0$. The trivial functions

$$(u(t), v(t)) = (u_*, v_*) \qquad (3.5)$$

is a (constant) orbit.

The pendulum equation is

$$\theta_{tt} = -\sin\theta \qquad (3.6)$$

(The constant L/g has been absorbed into the time scale).

Set

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \theta \\ \theta_t \end{pmatrix} \qquad (3.8)$$

so that (3.6) can be rewritten as

$$\begin{pmatrix} \theta \\ \theta_t \end{pmatrix}_t = \begin{pmatrix} \theta_t \\ -\sin\theta \end{pmatrix} \qquad (3.9)$$

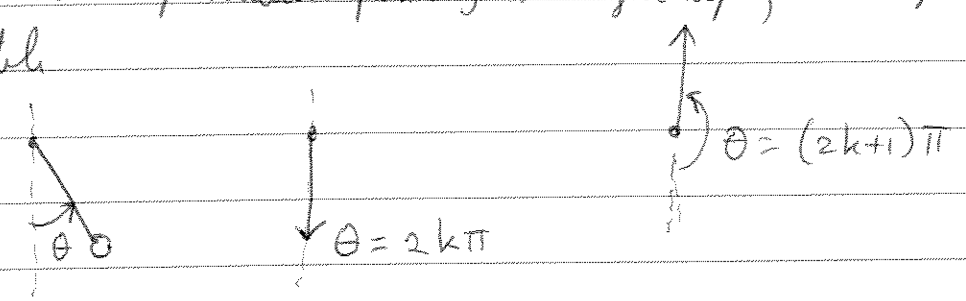
i.e.

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} v \\ -\sin u \end{pmatrix} \qquad (3.10)$$

The stationary points are

$$(\theta, \theta_t) = (n\pi, 0) \qquad (3.11)$$

The points with n even correspond to the pendulum hanging down. They are stable. The points with n odd correspond to the pendulum pointing straight up, and they are unstable.



Stability of Stationary Points

Let (u_*, v_*) be a stationary point for (3.4) and let

$$(u, v) = (u_* + u', \; v_* + v') \qquad (3.12)$$

Then

$$\begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix} = \begin{pmatrix} \cancel{f_*} + f_u u' + f_v v' + O(u'^2 + v'^2) \\ \cancel{g_*} + g_u u' + g_v v' + O(u'^2 + v'^2) \end{pmatrix}$$

in which $(f_*, g_*) = (f(u_*, v_*), \; g(u_*, v_*)) = 0$ and

$$M = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} = \begin{pmatrix} f_u(u_*, v_*) & f_v(u_*, v_*) \\ g_u(u_*, v_*) & g_v(u_*, v_*) \end{pmatrix} \qquad (3.13)$$

is a constant matrix. The linearized equations for (u', v') are

$$\begin{pmatrix} u' \\ v' \end{pmatrix}_t = M \begin{pmatrix} u' \\ v' \end{pmatrix} \qquad (3.14)$$

In most cases, the linearized equation is sufficient to describe the stability or instability of the stationary point (u_*, v_*)

Some definitions. Let $(u_*, v_*) = U_*$ be a stationary point

~~as points in~~

~~U_* is stable if all orbits starting in~~

- U_* is stable if for every ε there is δ so that all ~~points~~ orbits starting within distance δ of U_* stay within distance ε.
- U_* is asymptotically stable if there is a δ so that all orbits U starting within distance δ of have $\lim_{t \to \infty} U(t) = U_*$.

• U^* is marginally stable if it is stable but not asymptotically stable.

We now analyze the stability of a stationary point using the linearized equation (3.14). The solution to this equation is

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = e^{tM} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix} \qquad (3.15)$$

Let M have eigenvectors φ and ψ and eigenvalues λ and μ. Then the solution (3.15) can be rewritten as

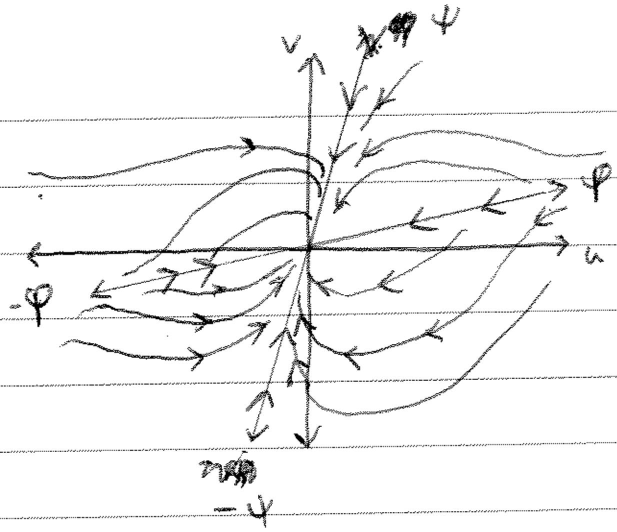
$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = a e^{\lambda t} \varphi + b e^{\mu t} \psi \qquad (3.16)$$

in which a, b are constant scalars, λ, μ are constant scalars, and φ, ψ are constant vectors. The stability can be characterized by the values of λ and μ.

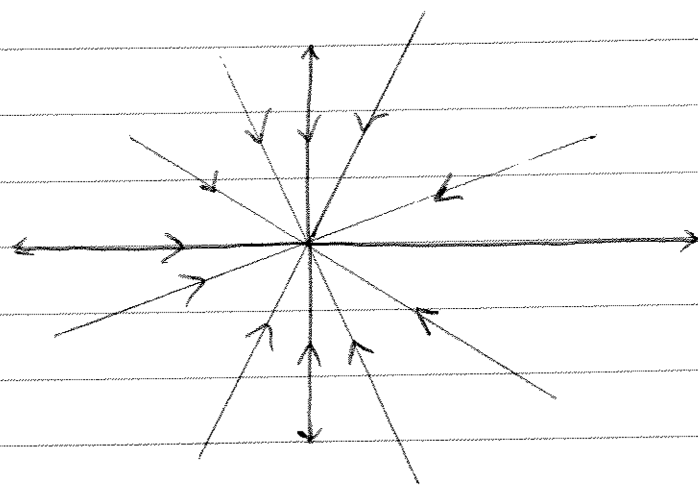
1. Stable node $\quad \lambda \le \mu < 0$.

In this case $(u', v') \to 0$ as $t \to \infty$, so that $(u, v) \to (u_*, v_*)$ as $t \to \infty$. Thus this is an asymptotically stable fixed point.

We also draw the phase plane pictures of the orbits for this system. There are four special orbits along along the directions $\pm\varphi$ and $\pm\psi$. If $\lambda = \mu$, then all orbits are straight lines through 0. If $\lambda < \mu$, then all orbits tend towards the ψ direction, since the φ component decays more rapidly than the ψ component



Stable node $\lambda < \mu$

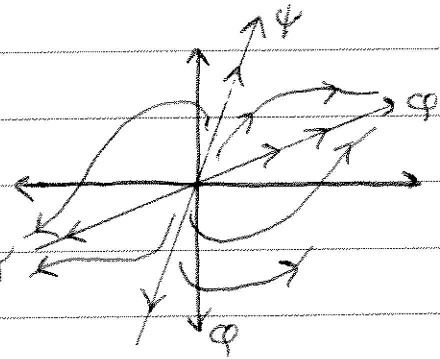


Stable node $\lambda = \mu$

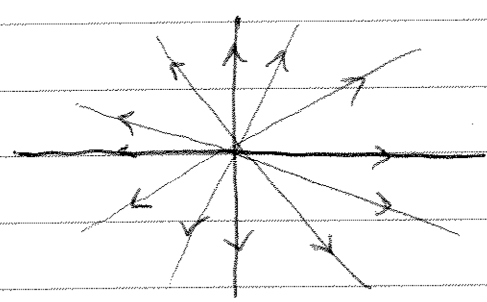
Note that if $\lambda = \mu$, then $M = \lambda I$.

2. Unstable node $0 < \mu \le \lambda$.

This is just the same as the stable node but with the direction of time reversed.



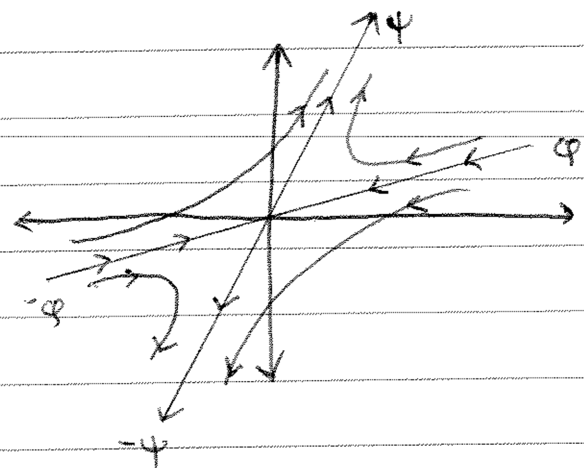
Unstable node $\mu < \lambda$



Unstable node $\mu = \lambda$

3. Saddle point $\lambda < 0 < \mu$

This has a single stable direction $\pm\varphi$. ~~Any solut~~ ~~with~~ There is also a special unstable solution growing in direction $\pm\psi$. All other orbits, with both $a \neq 0$ and $b \neq 0$ in (3.16) tend toward the unstable ψ direction.



Saddle point

~~4. Center $\lambda = \alpha i$, $\mu = -\alpha i$~~

The remaining nondegenerate (i.e. $\lambda \neq 0, \mu \neq 0$) types of fixed points involve complex e-values and e-vectors. If M is a real matrix ~~with~~ then

$$M\varphi = \lambda\varphi$$

implies

$$M\psi^* = \lambda^* \varphi^*$$

in which $*$ denotes complex conjugate. This shows that for M having a complex eigenvalue, the

$$\lambda = \lambda_R + i\lambda_I$$
$$\mu = \lambda^* = \lambda_R - i\lambda_I \qquad (3.17)$$
$$\psi = \varphi^*$$

A real solution (u, v) given by (3.16) has $b = a^*$, so that

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = a e^{\lambda t} \varphi + a^* e^{\lambda^* t} \varphi^* \qquad (3.18)$$

$$= e^{\lambda_R t} \left(\chi_1 \cos \lambda_I t + \chi_2 \sin \lambda_I t \right)$$

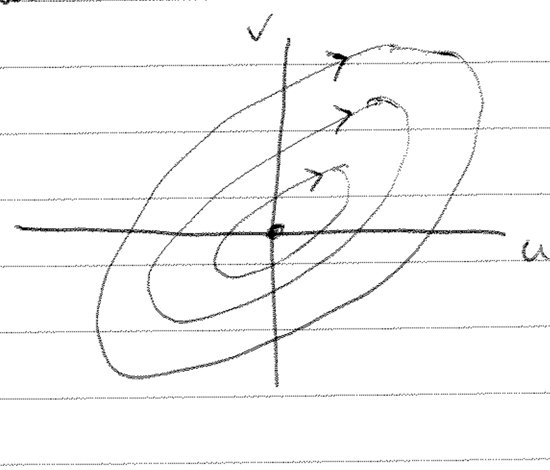
for some real vector χ_1 and χ_2.

4. <u>Center</u> $\lambda = i\lambda_I$, $\mu = -i\lambda_I$ (i.e. $\lambda_R = 0$)

In this case the solution has the form

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \chi_1 \cos \lambda_I t + \chi_2 \sin \lambda_I t$$

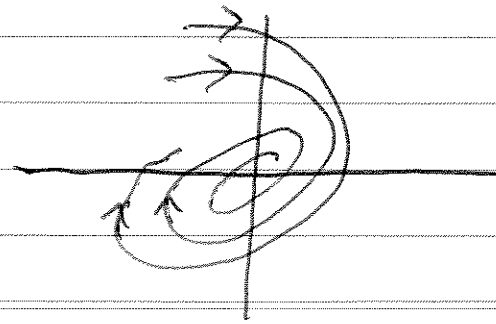
These solutions are all periodic with period $T = 2\pi/\lambda_I$.
One can show that the orbits are ellipses. In this case the linear equations have a marginally stable fixed point. Stability for the nonlinear equations will be determined by the nonlinear terms



Center

5. <u>Stable spiral</u> $\lambda_R < 0$

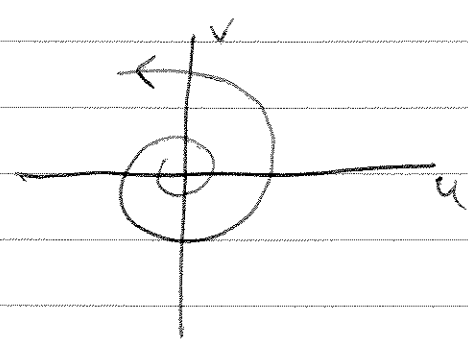
In this case, the solutions (3.18) decay but also rotate as t increases. The fixed point is stable



Stable spiral

6. <u>Unstable spiral</u> $\lambda_R < 0$

This is the opposite of the stable spiral. This fixed point is unstable



Unstable spiral