matrices that are <u>not diagonalizable</u>, i.e. matrices with

Next we consider matrices that are non-self-adjoint. If the matrix A is "normal," which is defined by $AA^+ = A^+A$, then the properties of the ODE $x' = Ax$ are more or less the same as for a self-adjoint matrix.

If A is not normal, then there are two phenomena that are not seen for self-adjoint matrices. The first is <u>generalized eigenvectors</u>. An example is

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad (2.64)$$

Then $A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

In general

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}$$

from which it follows that

$$A^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \qquad (2.65)$$

Now

$$\sum_{k=0}^{\infty} \frac{1}{k!} k t^k = t e^t \qquad (2.66)$$

$$e^{tA} = \sum \frac{t^k}{k!} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^t & t e^t \\ 0 & e^t \end{pmatrix} \tag{2.67}$$

The solution of

$$x' = Ax$$

$$x(0) = x_0 = \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix} \tag{2.68}$$

is

$$x(t) = e^{tA} x_0 = e^t \begin{pmatrix} x_{01} + t x_{02} \\ x_{02} \end{pmatrix} \tag{2.69}$$

This shows that for non-diagonalizable matrices, there is algebraic dependence, as well as exponential dependence, on t.

The second type is for matrices with non-orthogonal eigenvectors. If

$$A v_k = \lambda_k v_k \tag{2.70}$$

Then

$$x_k(t) = e^{t\lambda_k} v_k \tag{2.71}$$

is a solution of

$$x' = Ax \tag{2.72}$$

The properties of the solution are not as simple as for a self-adjoint matrix. Consider the example

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1+\varepsilon \end{pmatrix} \tag{2.73}$$

for which

$$\lambda_1 = -1 \qquad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = -1+\varepsilon \qquad v_2 = \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} \tag{2.74}$$

The general solution has the form

$$X(t) = a e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b e^{-(1-\varepsilon)t} \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix}$$

$$= \begin{pmatrix} (a + b e^{\varepsilon t}) e^{-t} \\ \varepsilon b e^{-(1-\varepsilon)t} \end{pmatrix} \tag{2.75}$$

Suppose that $x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ so that $b = \frac{1}{\varepsilon}$, $a = -b = -\frac{1}{\varepsilon}$.

Then

$$x(t) = \begin{pmatrix} \frac{1}{\varepsilon}(e^{\varepsilon t} - 1) e^{-t} \\ e^{-(1-\varepsilon)t} \end{pmatrix} \tag{2.76}$$

Although the eigenvalues are negative, the solution still grows for small t.

Finally consider $A = A(t)$. As a simple example let A be piecewise constant

$$x' = \begin{cases} Ax & 0 < x < 1 \\ Bx & 1 < x \end{cases} \tag{2.77}$$

$$x(0) = x_0$$

Then
$$x(t) = \begin{cases} e^{tA} x_0 & 0 < x < 1 \\ e^{(t-1)B} e^{A} x_0 & 1 < x \end{cases} \tag{2.78}$$

In particular $x(2) = e^{B} e^{A} x_0$. Unfortunately it is not generally true that $e^B e^A = e^{B+A}$. Expand out to quadratic term

$$\begin{aligned} e^B e^A &= (I + B + \tfrac{1}{2}B^2 + \dots)(I + A + \tfrac{1}{2}A^2 + \dots) \\ &= I + (B+A) + (\tfrac{1}{2}B^2 + \tfrac{1}{2}A^2 + BA) + \dots \end{aligned}$$

$$\begin{aligned} e^{B+A} &= I + (B+A) + \tfrac{1}{2}(B+A)^2 + \dots \\ &= I + (B+A) + \tfrac{1}{2}(B^2 + A^2 + BA + AB) + \dots \end{aligned}$$

so that

$$\begin{aligned} e^{B+A} - e^B e^A &= BA - \tfrac{1}{2}(BA + AB) \\ &= \tfrac{1}{2}(BA - AB) \\ &= \tfrac{1}{2}[B, A] \end{aligned} \tag{2.79}$$

in which $[B,A] = BA - AB$ is the <u>commutator</u> of B and A.

More generally if

$$X' = A(t)\,X$$
$$X(0) = X_0 \qquad (2.80)$$

then the solution ~~can be approximated by~~ cannot be written as a simple exponential. Set

$$a(t) = e^{\int_0^t A(s)\,ds} \qquad (2.81)$$

Then in general, $X(t) \neq \Phi(t)X_0$, and in fact it is not easy to write out $\frac{d}{dt}a(t)$, unless $[A(t_1), A(t_2)] = 0$ for all t_1 and t_2.