Solution for constant A

Consider the system

$$x' = Ax \qquad x(0) = x_0 \tag{2.46}$$

in which A is independent of t. Just as if A were a scalar the solution is given by

$$x(t) = e^{tA} x_0$$

in which exponential of a matrix is defined as

$$e^{B} = I + B + \tfrac{1}{2} B^2 + \tfrac{1}{6} B^3 + \dots$$
$$= \sum_{n=0}^{\infty} (n!)^{-1} B^n \tag{2.47}$$

Check that

$$\frac{d}{dt} e^{tA} = \frac{d}{dt} \sum_{n=0}^{\infty} (n!)^{-1} (tA)^n$$

$$= \frac{d}{dt} \sum_{n=0}^{\infty} (n!)^{-1} t^n A^n$$

$$= \sum_{n=0}^{\infty} (n!)^{-1} n\, t^{n-1} A^n$$

$$= \sum_{n=1}^{\infty} ((n-1)!)^{-1} t^{n-1} A^n$$

$$= A \sum_{n=1}^{\infty} ((n-1)!)^{-1} t^{n-1} A^n$$

$$= A \sum_{n=0}^{\infty} (n!)^{-1} t^n A^n$$

$$= A e^{tA} \tag{2.48}$$

Thus

$$\frac{dx}{dt} = \frac{d}{dt} e^{tA} x_0 = \hat{A}\left(e^{tA} x_0\right) = Ax \tag{2.49}$$

The exponential is especially easy to evaluate if A is self-adjoint. In this case

$$A = U \Lambda U^{+} \tag{2.50}$$

Calculate

$$\begin{aligned} A^2 &= (U\Lambda U^{+})(U\Lambda U^{+}) \\ &= U\Lambda (U^{+}U) \Lambda U^{+} \\ &= U\Lambda^2 U^{+} \end{aligned} \tag{2.51}$$

since $U^{+}U = I$. The same calculation works for A^k so that

$$A^k = U \Lambda^k U^{+} \tag{2.52}$$

It follows that

$$\begin{aligned} e^A &= \sum_{n=0}^{\infty} (n!)^{-1} A^n \\ &= \sum_{n=0}^{\infty} (n!)^{-1} U\Lambda^n U^{+} \\ &= U \sum_{n=0}^{\infty} (n!)^{-1} \Lambda^n U^{+} \\ &= U e^{\Lambda} U^{+} \end{aligned} \tag{2.53}$$

Moreover since $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, it follows that

$$\Lambda^2 = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{pmatrix}$$

$$= \mathrm{diag}(\lambda_1^2, \ldots, \lambda_n^2$$

and

$$\Lambda^n = \mathrm{diag}(\lambda_1^n, \ldots, \lambda_n^2) \tag{2.54}$$

It follows that

$$e^{\Lambda} = \mathrm{diag}\,(e^{\lambda_1}, \dots, e^{\lambda_n}) \qquad (2.55)$$

Combining (2.53) and (2.55) leads to

$$e^{A} = U\,\mathrm{diag}\,(e^{\lambda_1}, \dots, e^{\lambda_n})\, U^{+} \qquad (2.56)$$

and

$$e^{tA} = U\,\mathrm{diag}\,(e^{t\lambda_1}, \dots, e^{t\lambda_n})\, U^{+} \qquad (2.57)$$

This suggests that the solution of the system (2.46) can be reduced to the solution of n scalar ODE's. The easiest way to see this is to use the eigenvectors $v_1, \dots, v_n$ of A.
~~Take the dot~~ Since A is self-adjoint

$$A v_k = \lambda_k v_k$$
$$v_k^{+} A = \lambda_k v_k^{+} \qquad (2.58)$$

Now take the dot product of v_k with (2.46)

$$v_k^{+} x' = v_k^{+} A x \qquad v_k^{+} x(0) = v_k^{+} x_0 \qquad (2.59)$$

which becomes

$$(v_k^{+} x)' = \lambda_k v_k^{+} x \qquad (2.60)$$

~~Denote $\alpha_k = v_k^{+} x$, so that (2.60)~~ says

$$\cancel{\alpha_k' = \lambda_k \alpha_k}$$
$$\cancel{\alpha_k(0) = v_k^{+} x_0} \qquad \cancel{(2.61)}$$

The solution of (2.60) is

$$v_k^+ x(t) = e^{t\lambda_k} \, v_k^+ x_0 \, . \qquad (2.61)$$

This says that the scalar coefficients $\hat{x}_k(t) = v_k^+ x(t)$ evolve by

$$\hat{x}_k(t) = e^{t\lambda_k} \, \hat{x}_k \qquad (2.62)$$

These are coefficients for $x(t)$ in the sense that

$$x(t) = \sum_{k=1}^{n} \hat{x}_k(t) \, v_k \qquad (2.63)$$