6. Two Point Boundary Value Problems

An Example

There are classical problems of great significance for practical applications and mathematical theory. The prototypical problem is

$$u_{xx} = f(x) \qquad 0 < x < 1 \tag{6.1}$$
$$u(0) = u(1) = 0$$

The solution can be found in terms of a Green's function $G(x, y)$ or an eigenvector expansion.

The Green's function is

$$G(x,y) = \begin{cases} -(1-y)\,x & 0 < x < y < 1 \\ -\,y\,(1-x) & 0 < y < x < 1 \end{cases} \tag{6.2}$$

which satisfies

$$G(0,y) = G(1,y) = 0 \qquad 0 < y < 1$$
$$G(x, x+) = G(x, x-) \tag{6.3}$$
$$G_x(x, x+) - G_x(x, x-) = 1$$
$$G_{xx}(x,y) = 0$$

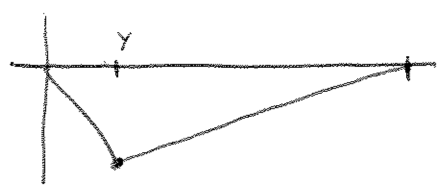

Fig 6.1 Plot of $G(x,y)$ vs x.

The solution of (6.1) is then

$$u(x) = \int_0^1 G(x,y) f(y)\,dy$$
$$= -\int_0^x y(1-x) f(y)\,dy - \int_x^1 (1-y)x f(y)\,dy \tag{6.4}$$

The eigenfunction for (6.1) are

$$\varphi_k = \sqrt{2} \sin k\pi x \qquad (6.5)$$
$$\lambda_k = -k^2\pi^2$$

satisfying

$$\varphi_{kxx} = \lambda_k \varphi_k \qquad 0 < x < 1 \qquad (6.6)$$
$$\varphi_k(0) = \varphi_k(1) = 0$$

These are orthonormal, i.e.

$$\int_0^1 \varphi_k(x)\, \varphi_\ell(x)\, dx = \delta_{k\ell} = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases} \qquad (6.7)$$

They are also complete, i.e. any function u satisfying

$$u(0) = u(1) = 0 \qquad (6.8)$$
$$\int_0^1 u_x^2\, dx < \infty$$

can be written as

$$u(x) = \sum_{k=1}^{\infty} a_k \varphi_k(x) \qquad (6.9)$$

$$a_k = (u, \varphi_k) \equiv \int_0^1 u(x)\varphi_k(x)\, dx \qquad (6.10)$$

in which the sum converges in the norm $\int_0^1 u^2 + u_x^2\, dx$.

Expand u and f as

$$u(x) = \sum_{k=1}^{\infty} a_k \varphi_k(x) \qquad (6.11)$$

$$f(x) = \sum_{k=1}^{\infty} b_k \varphi_k(x).$$

Then $u_{xx} = f$ becomes

$$u_{xx} = \sum_{k=1}^{\infty} \lambda_k a_k \varphi_k = \sum b_k \varphi_k \qquad (6.12)$$

so that

$$a_k = \frac{b_k}{\lambda_k} = -\frac{b_k}{\pi^2 k^2} \qquad (6.13)$$

<u>General System</u>

More generally consider a 2nd order operator

$$L u = p_0 u_{xx} + p_1 u_x + p_2 u \qquad a < x < b \qquad (6.14)$$

$$B_i u = a_{i1} u(a) + a_{i2} u(b) + a_{i3} u_x(a) + a_{i4} u_x(b) \qquad i = 1, 2$$

with $p_j = p_j(x)$, smooth functions and

$$p_0(x) \neq 0 \quad \forall \quad a < x < b. \qquad (6.15)$$

Define the inner product

$$(u, v) = \int_a^b u(x) \overline{v(x)}\, q(x) dx \qquad (6.16)$$

for u and v smooth complex functions, and q a smooth real, positive function. Require that L is self-adjoint with boundary operators B_1 and B_2; i.e. if

$$B_1 u = B_2 u = B_1 v = B_2 v = 0 \qquad (6.17)$$

then

$$(u, Lv) = (Lu, v). \qquad (6.18)$$

Under this assumption, the problem

$$\begin{aligned} Lu &= f \\ B_1 u = B_2 u &= 0 \end{aligned} \qquad (6.19)$$

has all of the properties described above for the problem (6.1).

For simplicity we take $p_0 \equiv 1$, [illegible] i.e.

$$Lu = u_{xx} + p_1(x)\,u_x + p_2(x)\,u \qquad (6.20)$$

and

$$\begin{aligned} B_1 u &= u(a) \\ B_2 u &= u(b) \end{aligned} \qquad (6.21)$$

so that the boundary conditions are

$$u(a) = u(b) = 0. \qquad (6.22)$$

Green's Function

In this section, we construct the Green's function $G(x,y)$ for the ~~problem~~ operator L from (6.20) with boundary conditions (6.22).

First define the delta function $\delta(x)$ and Heaviside function $H(x)$ by

$$H(x) = \begin{cases} 0 & x<0 \\ 1 & x>0 \end{cases} \qquad (6.23)$$

and for any continuous function φ

$$\int \delta(x)\,\varphi(x)\,dx = \varphi(0). \qquad (6.24)$$

It follows that

$$H(x-y) = \begin{cases} 0 & x<y \\ 1 & x>y \end{cases} \qquad (6.25)$$

$$\int \delta(x-y)\,\varphi(x)\,dx = \varphi(y) \qquad (6.26)$$

and

$$H_x(x) = \delta(x) \qquad (6.27)$$

Now we derive equations for G. For any smooth function f, the solution of

$$\begin{aligned} &Lu = f \qquad a<x<b \\ &u(a) = u(b) = 0 \end{aligned} \qquad (6.28)$$

should be given by

$$u(x) = \int_a^b G(x,y)\, f(y)\,dy \qquad (6.29)$$

Apply L to (6.29) to get for any f

$$\begin{aligned} Lu &= \int_a^b L_x G(x,y)\, f(y)\,dy = f(x) \\ \int_a^b G(a,y) f(y)\,dy &= \int_a^b G(b,y)\, f(y)\,dy \end{aligned} \qquad (6.30)$$

from which it follows that

$$\begin{aligned} L_x G(x,y) &= \delta(x-y) \qquad \forall\, x,y \in (a,b) \\ G(a,y) &= G(b,y) = 0 \end{aligned} \qquad (6.31)$$

The construction of G depends on an additional assumption: that the problem (6.28) has no nontrivial homogeneous solution, i.e. if

$$\begin{aligned} L\varphi_0 &= 0 \qquad a < x < b \\ \varphi_0(a) &= \varphi_0(b) = 0 \end{aligned} \qquad (6.32)$$

then

$$\varphi_0 \equiv 0 . \qquad (6.33)$$

Otherwise (if (6.32) has a solution $\varphi_0 \neq 0$), then take the inner product of (6.28) with φ_0 to get

$$(\varphi_0, f) = (\varphi_0, Lu) = (L\varphi_0, u) = 0 \qquad (6.34)$$

In this case (6.28) can be solved only if f satisfies (6.34) and the construction of G must be changed.

Under the condition (6.33), we now construct G. Let φ_1, φ_2 solve

$$L\varphi_i = 0 \qquad a < x < b \qquad (6.35)$$

with

$$\begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi_{1x} & \varphi_{2x} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{at } x=0 . \qquad (6.36)$$

These functions exist by solution of the initial value problem.

Define the Wronskian

$$W(y) = \varphi_1(y)\,\varphi_{2y}(y) - \varphi_2(y)\,\varphi_{1y}(y) \qquad (6.37)$$

with

$$W(a) = 1$$

$$\begin{aligned} W_y &= \varphi_1 \varphi_{2yy} - \varphi_2 \varphi_{1yy} \\ &= -\varphi_1(p_1 \varphi_{2y} + p_2 \varphi_2) + \varphi_2(p_1 \varphi_{1y} + p_2 \varphi_1) \\ &= -p_1(\varphi_1 \varphi_{2y} - \varphi_2 \varphi_{1y}) \\ &= -p_1 W \end{aligned} \qquad (6.38)$$

so that

$$W(y) = \exp\left(-\int_a^y p(y')\,dy'\right)$$
$$\neq 0 \qquad \forall y . \qquad (6.39)$$

Set

$$K(x,y) = \begin{cases} \frac{1}{W(y)} \left(\varphi_1(y)\varphi_2(x) - \varphi_2(y)\varphi_1(x) \right) & a<y<x \\ 0 & a<x<y<b \end{cases} \tag{6.40}$$

This satisfies

$$L_x K = 0 \qquad x \neq y \tag{6.41}$$

$$K(y+,y) = K(y-,y) = 0 \tag{6.42}$$

and

$$K_x = \begin{cases} \frac{1}{W(y)} \left(\varphi_1(y)\varphi_{2x}(x) - \varphi_2(y)\varphi_{1x}(x) \right) \\ 0 \end{cases} \tag{6.43}$$

so that

$$\begin{aligned} K_x(y+,y) &= 1 \\ K_x(y-,y) &= 0 \end{aligned} \tag{6.44}$$

Since K is continuous in x and its first derivative has a jump of size 1, then ~~this shows that~~ for y near x, $K_{xx} \cong \delta(x-y)$. In fact

$$L_x K(x,y) = \delta(x-y). \tag{6.45}$$

In order to satisfy the boundary conditions, set

$$G(x,y) = K(x,y) + \alpha(y)\varphi_1(x) + \beta(y)\varphi_2(x) \tag{6.46}$$

so that

$$\begin{aligned} G(a,y) &= 0 \\ G(b,y) &= 0 \end{aligned} \tag{6.47}$$

This is a linear system for α and β; i.e.

$$\begin{pmatrix} \varphi_1(a) & \varphi_2(a) \\ \varphi_1(b) & \varphi_2(b) \end{pmatrix} \begin{pmatrix} \alpha(y) \\ \beta(y) \end{pmatrix} = -\begin{pmatrix} K(a,y) \\ K(b,y) \end{pmatrix} \tag{6.48}$$

This is solveable, provided that

$$\varphi_1(a)\,\varphi_2(b) - \varphi_2(a)\,\varphi_1(b) \neq 0. \tag{6.49}$$

If this condition is violated, then there are constants α_0, β_0 with

$$\begin{pmatrix} \varphi_1(a) & \varphi_2(a) \\ \varphi_1(b) & \varphi_2(b) \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.50}$$

so that the function

$$\varphi_0(x) = \alpha_0\,\varphi_1(x) + \beta_0\,\varphi_2(x) \tag{6.51}$$

satisfies

$$L\varphi_0 = 0 \tag{6.52}$$
$$\varphi_0(a) = \varphi_0(b) = 0$$

in contradiction with the assumption (6.33). (This is the only place that (6.33) is used).

Now that the Green's function $G(x,y)$ has been constructed, we can define the Green's operator $\mathcal{G}$ by

$$\mathcal{G}f(x) = \int_a^b G(x,y)\,f(y)\,dy \tag{6.53}$$

Then

$$L\mathcal{G}f = f \quad \text{and} \quad \mathcal{G}Lu = u \tag{6.54}$$

i.e.

$$\mathcal{G} = L^{-1} \tag{6.55}$$

In particular $\mathcal{G}$ is self-adjoint and G is continuous with bounded derivative.

Self-adjointness of $\mathcal{G}$ implies that

$$G(x,y) = G(y,x) \tag{6.56}$$

<u>Eigenfunctions and Eigenvalues</u>

Consider the eigenfunction problem

$$\cancel{L\varphi = \lambda\varphi}$$
$$\cancel{B_1\varphi = B_2\varphi = 0}$$

$$\begin{aligned} L\psi &= \lambda\psi \\ B_1\psi &= B_2\psi = 0 \end{aligned} \tag{6.57}$$

Set $\varphi = L\psi$. Then since $\mathcal{G} = L^{-1}$,

$$\mathcal{G}\varphi = \mu\varphi \tag{6.58}$$

with

$$\mu = \lambda^{-1} \tag{6.59}$$

An important advantage of using $\mathcal{G}$ rather than L is that $\mathcal{G}$ is a bounded operator, whereas L is unbounded. Also, the boundary conditions are built into $\mathcal{G}$.

The problem (6.58) will be solved by a variational principle. Define the Rayleigh-Ritz quotient as

$$\frac{(u, \mathcal{G}u)}{(u,u)} \tag{6.60}$$

We will show that

$$\sup_{u \in L^2} \frac{|(u, \mathcal{G}u)|}{(u,u)} = \sup_{\|u\|=1} |(u, \mathcal{G}u)| \equiv |||\mathcal{G}||| \quad (6.61)$$

yields an eigenvalue. This is

<u>Theorem 6.1</u> There is an eigenvalue μ for $\mathcal{G}$ with

$$\mu = |||\mathcal{G}||| \quad \text{or} \quad \mu = -|||\mathcal{G}||| \quad (6.62)$$

The proof of this theorem requires several technical results first.

<u>Proposition 6.2</u> <u>Arzela-Ascoli Theorem</u>

Let u_n be uniformly bounded and equi-continuous, i.e. there exists c independent of n and x such that

$$|u_n(x)| < c \quad (6.63)$$

and for all ε $\exists$ $\delta(\varepsilon)$ independent of x, x' and n such that if $|x-x'| < \delta(\varepsilon)$, then $|u_n(x) - u_n(x')| < \varepsilon$ (6.64)

Then there exists a subsequence u_{n_i} that is uniformly convergent

$$u_{n_i} \to u \qquad i \to \infty. \quad (6.65)$$

<u>Proposition 6.3</u>

If u_n satisfy $\|u_n\| \le c$, then $\mathcal{G}u_n$ are uniformly bounded and equicontinuous

Proof of Prop 6.3

Since $|G| < C$ is bounded then

$$|\mathcal{G} u_n| = \left| \int G u_n \, dy \right|$$

$$\le C \int_a^b u_n \, dy$$

$$\le C \left(\int_a^b u_n^2 \, dy \right)^{1/2} \left(\int_a^b 1 \, dy \right)^{1/2}$$

$$\le C \, (b-a)^{1/2} \, \| u_n \|$$

$$\le C^2 \, (b-a)^{1/2} \qquad (6.66)$$

Next let $\varepsilon > 0$ and let Δx be such that $|G(x+\Delta x) - G(x)| < \varepsilon$. Then

$$|\mathcal{G} u_n(x+\Delta x) - \mathcal{G} u_n(x)| \le \int_a^b |G(x+\Delta x, y) - G(x,y)| \, |u_n(y)| dy$$

$$\le \varepsilon \int_a^b |u_n| \, dy$$

$$\le \varepsilon \, (b-a)^{1/2} \, C^2 \qquad (6.67)$$

which finishes the proof.

Now define a different norm on $\mathcal{G}$ than (6.61).

<u>Definition</u> The operator norm is

$$\|\mathcal{G}\| = \sup_{\|u\|=1} \|\mathcal{G}u\| \qquad (6.68)$$

<u>Proposition 6.4</u> The two norms (6.61) and (6.68) are equal, i.e.

$$|||\mathcal{G}||| = \|\mathcal{G}\| \qquad (6.69)$$

i.e.

$$\sup_{\|u\|=1} |(u, \mathcal{G}u)| = \sup_{\|u\|=1} (\mathcal{G}u, \mathcal{G}u)^{1/2} \qquad (6.70)$$

<u>Proof of Prop 6.4</u>

Since for $\|u\|=1$

$$\begin{aligned} |(u, \mathcal{G}u)| &\leq \|u\| \, \|\mathcal{G}u\| \\ &\leq \|\mathcal{G}\| \end{aligned} \qquad (6.71)$$

Then

$$|||\mathcal{G}||| \leq \|\mathcal{G}\| \qquad (6.72)$$

Next show the reverse inequality. For any u and v

$$\begin{aligned} (\mathcal{G}u,u) + (\mathcal{G}v,v) + 2(\mathcal{G}u,v) &= (\mathcal{G}(u+v), (u+v)) \\ &\leq |||\mathcal{G}||| \, \|u+v\|^2 \end{aligned} \qquad (6.73)$$

$$\begin{aligned} (\mathcal{G}u,u) + (\mathcal{G}v,v) - 2(\mathcal{G}u,v) &= (\mathcal{G}(u-v), (u-v)) \\ &\geq -|||\mathcal{G}||| \, \|u-v\|^2 \end{aligned} \qquad (6.74)$$

Subtract these to get

$$4\,(Gu, v) \le |||G|||\,\left(\|u+v\|^2 + \|u-v\|^2\right)$$
$$= 2\,|||G|||\,\left(\|u\|^2 + \|v\|^2\right) \qquad (6.75)$$

Set $v = Gu/\|Gu\|$ with $\|u\|=1$ and $\|v\|=1$. The left hand side is

$$4\,(Gu, Gu)/\|Gu\| = 4\,\|Gu\|$$

and the right hand side is
$$4\,|||G|||$$
It follows that (after dividing by 4) for any $\|u\|=1$,

$$\|Gu\| \le |||G||| \qquad (6.76)$$

Thus
$$\|G\| = \sup_{\|u\|=1} \|Gu\| \le |||G||| \qquad (6.77)$$

Combine this with (6.72) to get
$$\|G\| = |||G|||. \qquad (6.78)$$
which finishes the proof.

Now we can prove Theorem 6.1

Proof of Theorem 6.1

Let u_n be a sequence of functions such that

$$\|u_n\| = 1$$
$$(\mathcal{G}u_n, u_n) \to |||\mathcal{G}||| \quad \text{or} \quad (\mathcal{G}u_n, u_n) \to -|||\mathcal{G}||| \tag{6.79}$$

For simplicity assume that the sign is + and set $\mu = |||\mathcal{G}|||$. By Proposition 6.3, $\mathcal{G}u_n$ is uniformly bounded and equicontinuous and by Proposition 6.2,

$$\mathcal{G}u_n \to \varphi_0 \quad \text{as } n \to \infty \tag{6.80}$$

uniformly on $[a,b]$ after replacement of u_n by a subsequence (if necessary).

It follows that

$$\|\mathcal{G}u_n - \varphi_0\| \to 0 \quad \text{as } n \to \infty$$
$$\|\mathcal{G}u_n\| \to \|\varphi_0\| \tag{6.81}$$

Thus

$$\|\mathcal{G}u_n - \mu u_n\|^2 = \|\mathcal{G}u_n\|^2 + \mu^2\|u_n\|^2 - 2\mu(\mathcal{G}u_n, u_n)$$
$$\to \|\varphi_0\|^2 + \mu^2 - 2\mu|||\mathcal{G}|||$$
$$= \|\varphi_0\|^2 - \mu^2 \tag{6.82}$$

This must be non-negative, so that

$$\|\varphi_0\|^2 \geq \mu^2 > 0 \tag{6.83}$$

Thus $\varphi_0 \neq 0$.

QED

also

$$\| \mathcal{G} u_n \|^2 \leq \| \mathcal{G} \|^2 = |||\mathcal{G}|||^2 = \mu^2 \tag{6.84}$$

so that

$$\begin{aligned} 0 \leq \| \mathcal{G} u_n - \mu u_n \|^2 &= \| \mathcal{G} u_n \|^2 + \mu^2 \| u_n \|^2 - 2\mu (\mathcal{G} u_n, u_n) \\ &\leq 2\mu^2 - 2\mu (\mathcal{G} u_n, u_n) \\ &\longrightarrow 0 \qquad \text{as } n \to \infty \end{aligned}$$

by (6.79) and $\mu = |||\mathcal{G}|||$. Thus

$$\| \mathcal{G} u_n - \mu u_n \| \longrightarrow 0 \tag{6.85}$$

It follows that

$$\begin{aligned} \mathcal{G} \varphi_0 - \mu \varphi_0 &= \lim_{n \to \infty} \mathcal{G} (\mathcal{G} u_n - \mu u_n) \\ &= 0 \end{aligned} \tag{6.86}$$

Therefore φ_0 is an eigenfunction with eigenvalue μ. This finishes the proof of Theorem 6.1.

Additional Eigenfunctions

(in absolute value)

The eigenvalue $\mu_0 = \pm \| \mathcal{G} \|$ is the largest eigenvalue. To find the next largest eigenvalue, use the fact that the eigenfunctions for a self-adjoint operator are orthogonal, i.e. if

$$\begin{aligned} \mathcal{G} \varphi_0 &= \mu_0 \varphi_0 \\ \mathcal{G} \varphi_1 &= \mu_1 \varphi_1 \end{aligned} \tag{6.87}$$

with $\mu_0 \neq \mu_1$, then

$$(\varphi_0, \varphi_1) = 0 \tag{6.88}$$

If $\mu_0 = \mu_1$, but φ_0 and φ_1 are linearly independent, then we can always make φ_1 orthogonal to φ_0 by a linear combination of φ_0 and φ_1; i.e. for $\|\varphi_0\| = 1$

$$\tilde{\varphi}_1 = \varphi_1 - (\varphi_0, \varphi_1)\varphi_0 \tag{6.89}.$$

Now we use this fact and subtract off the φ_0 part of $\mathcal{G}$. Set $\|\varphi_0\| = 1$ and define

$$\begin{aligned} G_1(x,y) &= G(x,y) - \mu_0 \varphi_0(x) \varphi_0(y) \\ \mathcal{G}_1 &= \mathcal{G} - \mu_0 (\varphi_0, \cdot)\varphi_0 . \end{aligned} \tag{6.90}$$

Find φ_1 as in Theorem 6.1 by maximizing

$$\frac{|(\mathcal{G}_1 u, u)|}{\|u\|^2} = |||\mathcal{G}_1||| \tag{6.91}$$

and obtaining

$$\mu_1 = \pm |||\mathcal{G}_1||| \tag{6.92}$$

with an eigenfunction φ_1 satisfying

$$\mathcal{G}_1 \varphi_1 = \mu_1 \varphi_1 \qquad (6.93)$$

Now ~~$\mathcal{G}_1\varphi_0$~~

$$\begin{aligned}\mathcal{G}_1 \varphi_0 &= \mathcal{G}\varphi_0 - \mu_0 (\varphi_0, \varphi_0)\varphi_0 \\ &= \mu_0\varphi_0 - \mu_0 \varphi_0 \\ &= 0 \end{aligned} \qquad (6.93)$$

so that

$$\begin{aligned}(\varphi_0, \varphi_1) &= \mu_1^{-1} (\varphi_0, \mathcal{G}_1 \varphi_1) \\ &= \mu_1^{-1} (\mathcal{G}_1 \varphi_0, \varphi_1) \\ &= 0 \end{aligned} \qquad (6.94)$$

It follows that

$$\mathcal{G}_1 \varphi_1 = \mathcal{G} \varphi_1 \qquad (6.95)$$

so that

$$\mathcal{G} \varphi_1 = \mu_1 \varphi_1 . \qquad (6.96)$$

Thus we have found the next eigenfunction and eigenvalue.

This can be continued sequentially. For the k-th eigenfunction define

~~and~~ with

$$\begin{aligned}\mathcal{G}_k &= \mathcal{G}_{k-1} - \mu_{k-1} (\varphi_{k-1}, \cdot)\varphi_k \\ &= \mathcal{G} - \sum_{j=0}^{k-1} \mu_j (\varphi_j, \cdot) \varphi_j \end{aligned} \qquad (6.97)$$

Then μ_k, φ_k are found ~~from~~ as

$$\mu_k = \pm ||| \mathcal{G}_k ||| = \pm \sup_u \frac{|(\mathcal{G}_k u, u)|}{\|u\|^2} \qquad (6.98)$$

as in Theorem 6.1

An alternative is the following:

$$\mu_k = \sup_{u \perp \{\varphi_0, \dots, \varphi_{k-1}\}} \frac{|(G_k u, u)|}{\|u\|^2} \qquad (6.99)$$

and φ_k is the function that achieves this sup.

A second alternative is even more powerful, since it does not use the eigenfunctions at all

$$\mu_k = \inf_{u_0, \dots, u_{k-1}} \sup_{u \perp \{u_0, \dots, u_{k-1}\}} \frac{|(Gu, u)|}{\|u\|^2} \qquad (6.100)$$

This process terminates only if $\|G_m\| = 0$ for some m. In that case,

$$G f = \sum_{j=0}^{m-1} \mu_j \, (f, \varphi_j) \, \varphi_j \qquad (6.101)$$

and

$$L u = \sum_{j=0}^{m-1} \mu_j^{-1} \, (u, \varphi_j) \, \varphi_j \qquad (6.102)$$

But then

$$L u = 0 \qquad (6.103)$$

for all $u \perp \{\varphi_0, \dots, \varphi_{m-1}\}$ which was not allowed by assumption.

Properties of Eigenvalues and Eigenfunctions   The eigenfunction $\{\phi_k\}$ have many of the properties of the harmonic function $\sin \pi k x$ on $[0,1]$.

Bessel's inequality   For $f \in L^2(0,1)$ and ϕ_k an orthonormal sequence on $[0,1]$, then

$$\sum_{k=0}^{\infty} |(f, \phi_k)|^2 \leq \|f\|^2$$

This result is independent of the 2 point BVP, as is the next result

Parseval's identity   For $f \in L^2(0,1)$ and ϕ_k a complete orthonormal sequence on $[0,1]$, then

$$\sum_{k=0}^{\infty} |(f, \phi_k)|^2 = \|f\|^2.$$

A sequence is complete if $f = \sum_{k=0}^{\infty} a_k \phi_k$ for some a_k and for any f, with convergence in L^2.

Now we state properties of the sequence of eigenfunctions for a 2 pt BVP as discussed above.

Theorem 6.5   If $f \in C^2$ and f satisfies the boundary conditions, then

$$f = \sum_{0}^{\infty} (f, \phi_k) \phi_k$$

with uniform convergence.

Theorem 6.6   If $f \in L^2(0,1)$, then

$$f = \sum_{0}^{\infty} (f, \phi_k) \phi_k$$

with convergence in L^2.

Theorem 6.7 (Mercer's Theorem)

$$G(x,y) = \sum_{k=0}^{\infty} \mu_k \phi_k(x)\, \phi_k(y)$$

with uniform convergence.