5.3 Lorenz Equations

The Lorenz equations are

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned} \qquad (5.19)$$

They were derived by Ed Lorenz (1963) as a simplified model for atmospheric dynamics. The parameters σ, r, b are all positive. σ is the Prandtl number = viscosity / thermal diffusion. r is the Rayleigh number, a dimensionless number measuring the strength of convection. b is a measure of the aspect ratio of convective rolls in the atmosphere.

Two properties of this system constrain its flow. First is the symmetry $(x,y) \to (-x,-y)$; i.e. if $(x(t), y(t), z(t))$ is a solution then so is $(-x(t), -y(t), z(t))$.

The second property is volume contraction. Let V_0 be a set of initial data and let $V(t)$ be the resulting set at time t

$$V(t) = \{(x(t), y(t), z(t)) \text{ for } (x(0), y(0), z(0)) \in V_0\} \qquad (5.20)$$

If $\underset{\sim}{x} = (x,y,z)$ satisfies $\dot{\underset{\sim}{x}} = \underset{\sim}{f}$, then the normal velocity of the boundary of $V(t)$ is $\underset{\sim}{n} \cdot \underset{\sim}{f}$ in which $\underset{\sim}{n}$ is the outward normal to V. In a time dt, the volume swept out by an area dA of ∂V, is $(f \cdot n)\, dt\, dA$.

Figure 5.16

Let $V(t) =$ volume $(\mathcal{V}(t))$

The change in volume of $\mathcal{V}$ is then

$$V(t+dt) = V(t) + \int f\cdot n \; dA \, dt$$

i.e.

$$\frac{d}{dt} V(t) = \int_{\partial \mathcal{V}} f\cdot n \, dA$$

$$= \int_{\mathcal{V}} \nabla\cdot f \, dx \qquad (5.21)$$

For the Lorenz system

$$\nabla\cdot f = \partial_x(\sigma(y-x)) + \partial_y(rx-y-xz) + \partial_z(xy-bz)$$

$$= -\sigma - 1 - b$$

$$< 0 \qquad (5.22)$$

So

$$\frac{d}{dt} V \leq -(\sigma+1+b) \int_{\mathcal{V}} dx$$

$$= -(\sigma+1+b) V \qquad (5.23)$$

Thus $V(t)$ shrinks at an exponential rate.

Since the volume shrinks as time progresses, there cannot be any stationary points or closed orbits that are fully unstable. There also cannot be any ~~volumes that~~ 3D regions that are invariant under the flow. This shows that the stationary points must be saddles or stable nodes, stable spirals, or ~~stable~~ other types with at least one stable direction.

The stationary points satisfy

$$x - y = 0$$

$$rx - y - xz = 0$$

$$xy - bz = 0$$

Use the first equation in the third to get $z = x^2/b$, then use this in the second equation to get

$$b(r-1)x - x^3 = 0 \tag{5.24}$$

so that the stationary points are (x, y, z) satisfying

~~$(x,y,z) = (0,0,0)$~~

~~(x,y,z)~~

$$\text{(i)} \qquad x = y = z = 0 \tag{5.25}$$

$$\text{(ii)} \qquad x = y = \pm\sqrt{b(r-1)} \,, \quad z = r-1 \tag{5.26}$$

Lorenz named the second set of stationary points C^+ and C^-. As $r \to 1^+$, C^+ and C^- coallesce into the origin in a pitchfork bifurcation

For the fixed point 0, the linearized equation is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}^{\cdot} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{5.27}$$

~~and which the eigenvalues satisfy~~

One eigenvalue is $\lambda_3 = -b$, which is always stable. The other two eigenvalues come from the 2×2 upper left part of the matrix ~~with~~ $A_{22} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix}$ with

$$\lambda_1 + \lambda_2 = \text{trace}(A) = -(\sigma+1) \qquad (5.28)$$

$$\lambda_1 \lambda_2 = \det(A) = \sigma(r-1)$$

The discriminant is $\text{trace}^2 - 4\det > 0$, so that all eigenvalues are real.
For $r<1$, $\lambda_1\lambda_2 > 0$ and $\lambda_1+\lambda_2<0$ so that $\lambda_1<0$ and $\lambda_2<0$.
For $r>1$, $\lambda_1\lambda_2 > 0$ and $\lambda_1+\lambda_2<0$ so that $\lambda_1>0$ and $\lambda_2<0$
and

For $r<1$, the fixed point O is a stable node, while for $r>1$ it is a saddle. Moreover, for $r<1$, one can show that all orbits approach O as $t\to\infty$.

Consider
$$V = \frac{1}{\sigma}x^2 + y^2 + z^2 \qquad (5.29)$$

$$\begin{aligned}
\tfrac{1}{2}\dot V &= \tfrac{1}{\sigma}x\dot x + y\dot y + z\dot z \\
&= (yx - x^2) + (ryx - y^2 - xyz) + (xyz - bz^2) \\
&= (r+1)xy - x^2 - y^2 - bz^2 \\
&= -\left(x - \tfrac{r+1}{2}y\right)^2 - \left(1 - \left(\tfrac{r+1}{2}\right)^2\right)y^2 - bz^2 \\
&\le 0 \qquad (5.30)
\end{aligned}$$

if $r<1$ with equality iff $x=y=z$. This shows that on the orbit $(x(t), y(t), z(t))$, V continually decreases. It follows that

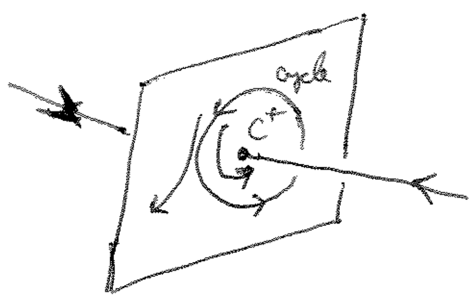
$$\lim_{t\to 0}\,(x(t), y(t), z(t)) = 0 \qquad (5.31)$$

for $r<1$.

For $r > 1$, both C^+ and C^- exist. A direct calculation shows that they are stable for

$$1 < r < r_H = \sigma(\sigma+b+3)/(\sigma-b-1) \qquad (5.32)$$

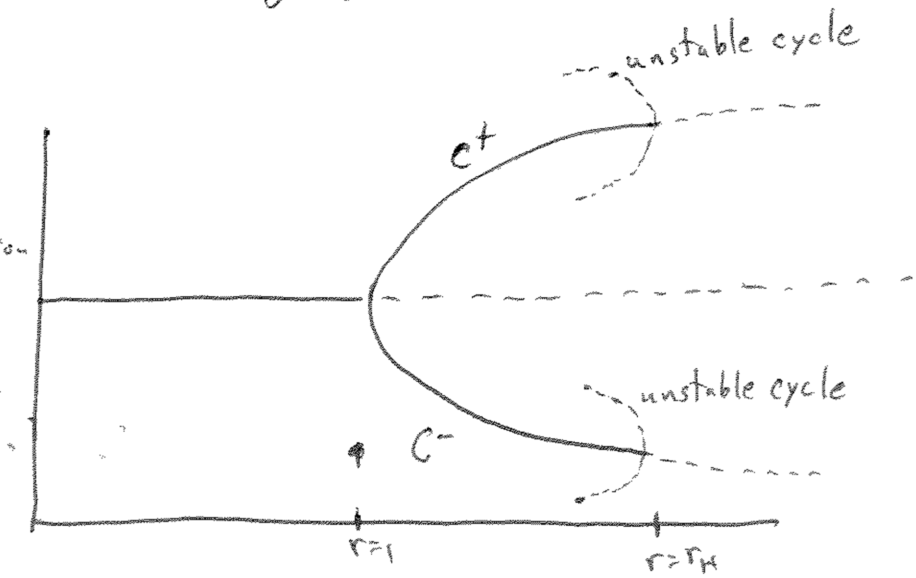
assuming that $\sigma > b+1$. At $r = r_H$ there is a Hopf bifurcation that is subcritical. ~~[illegible]~~ For r slightly less than r_H, ($r < r_H$), there is a small unstable limit cycle. For r slightly larger than r_H ($r > r_H$), the points C^+ and C^- become unstable and there are no limit cycles. These properties are indicated in the following figures



Phase plane for flow near fixed point C^+, showing unstable limit cycle, and directions that flow into C^+.

Figure 5.17

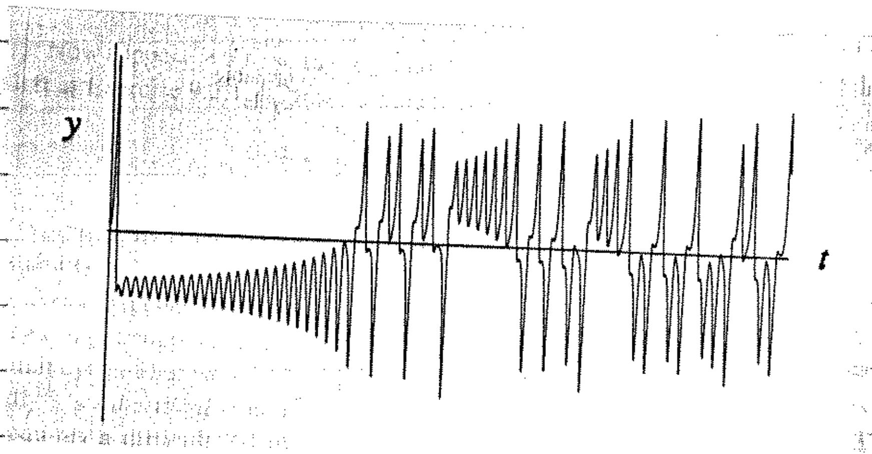
Bifurcation diagram for Lorenz equation Fig 5.18



Strange Attractor

One can also show that all trajectories remain bounded. Lorenz also argued that there are no stable limit cycles. So the orbits must move around in phase space, repelled by one object after another.

Most of the remaining results come from numerical computation. Lorenz studied $\sigma = 10$, $b = 8/3$, $r = 28$, with $r_H = 24.74$. A computation of $y(t)$ is shown in Figure 5.19.

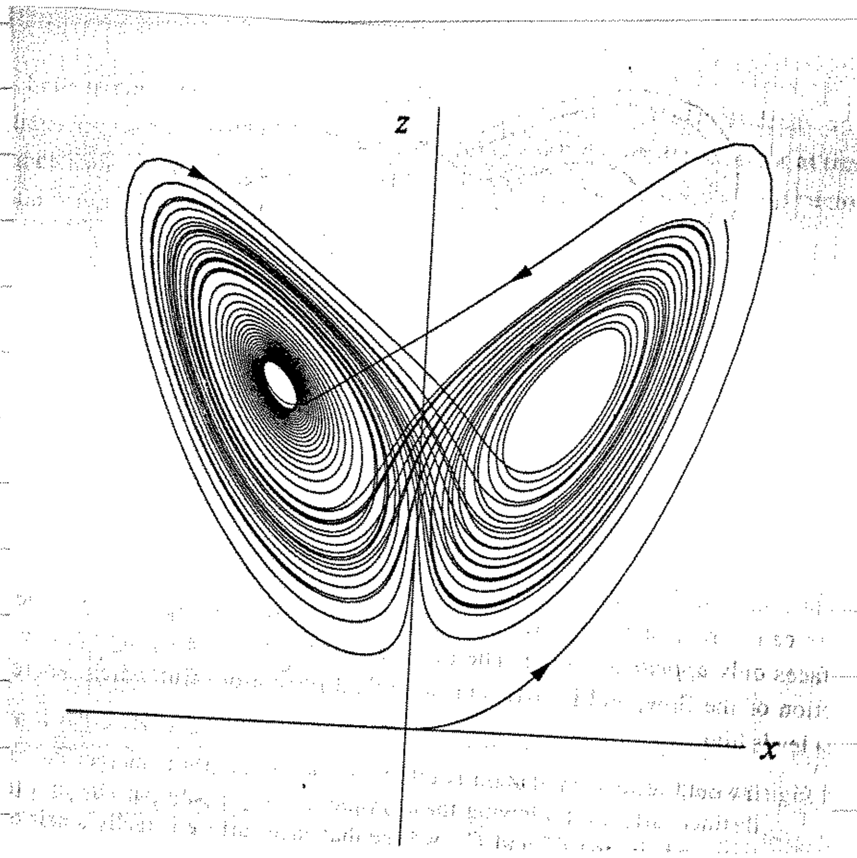


y vs. t. from Lorenz with
$\sigma = 10$, $b = 8/3$, $r = 28$,
from Strogatz

Fig. 5.19

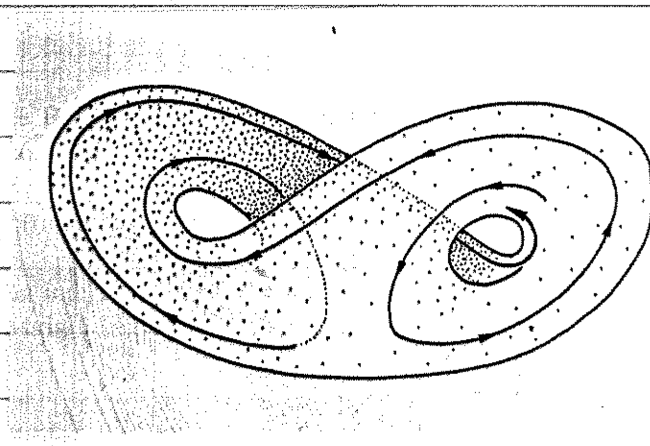
This figure shows that that orbit settles into an irregular oscillation that persists out to $t \to \infty$, but is aperiodic.

A different plot of (x, z), shows a butterfly shaped for this 2D projection of the orbits. The orbit starts on the right, then swings around to the left. It then does some number of loops on the left and returns to the right. This is repeated in an irregular fashion.



Projection of the orbits onto the (x, z) plane.
Fig. 5.20. from Strogatz.

In 3 dimensions the orbits are attracted to a very thin set that looks like a pair of butterfly wings. This set, shown schematically in Fig. 5.21, is called a strange attractor. In particular this set is not smooth. Rather it ~~has~~ is a fractal, with infinite surface area but 0 volume.



3D Strange attractor for the system of Figs 5.19
from Abraham and Shaw

Fig 5.21

Chaos

The motion on (or near) the attractor has sensitive dependence on initial data. This means that 2 points starting close together on the attractor will rapidly diverge from each other. In fact a small blob of initial data would eventually spread out over the entire attractor.

To measure this spread-out quantitatively consider two orbits $\underset{\sim}{x}_1(t)$ and $\underset{\sim}{x}_2(t)$ and set

$$\underset{\sim}{\delta}(t) = \underset{\sim}{x}_2(t) - \underset{\sim}{x}_1(t). \qquad (5.33)$$

One finds that

$$\|\delta(t)\| \sim \|\delta_0\| \, e^{\lambda t} \qquad (5.34)$$

with $\delta = 0.9$. This shows exponential divergence of trajectories.

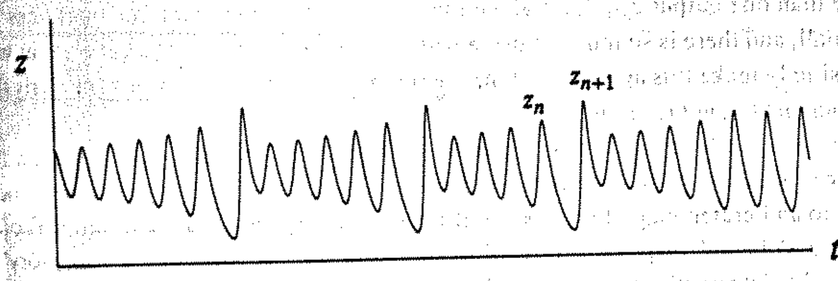
The number λ is called a <u>Lorenz exponent</u>. For any two given orbits, the number $\delta(t)$ eventually stops growing since the attractor is bounded. Also the value of $||\delta(t)||$ doesn't smoothly grow. These problems are fixed by taking the value $||\delta(0)||$ to be smaller and smaller.

A working definition of chaos is the following (from Strogatz)
Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.

"Aperiodic long term behavior" rules out trajectories that settle down to fixed points, periodic orbits or quasi-periodic orbits. This also does not allow trajectories that go off to ∞, since ∞ can then be thought of as a fixed point that is attracting.
"Deterministic" means that the strange behavior is not caused by statistical effects.

Lorenz map

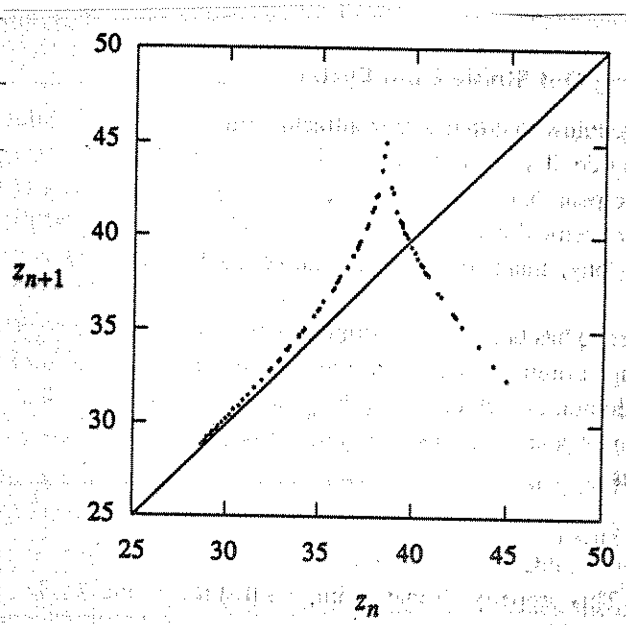
Lorenz found a simple method to analyze the dynamics on the strange attractor. Consider the plot of z vs. t and denote the value of the n-th local maximum as z_n. Lorenz suggested that z_{n+1} should be determined by z_n.



z vs. t from Strogatz

Figure 5.22

In order to check whether $z_{n+1} = f(z_n)$, he plotted z_{n+1} vs z_n as follows.



z_{n+1} vs. z_n

Figure 5.23

The result in Figure 5.23 shows that the points (z_n, z_{n+1}) (nearly) fall on a curve $(z, f(z))$. The function f is called the Lorenz map. Although the set of points (z_n, z_{n+1}) does have some finite thickness (i.e. there can be more than one value of z_{n+1} for a given value of z_n), the thickness is quite small.