5. Dynamical Systems

5.1 Bifurcations of First Order Autonomous Equations

Here we study the transitions between different solutions type, i.e. bifurcations, for first order systems

$$\dot{x} = f(x, r) \tag{5.1}$$

as a parameter r is varied.

The classification is simple for first order equations, but doesn't change too much for higher order.

Saddle-node bifurcation Consider

$$\dot{x} = r + x^2 \tag{5.2}$$

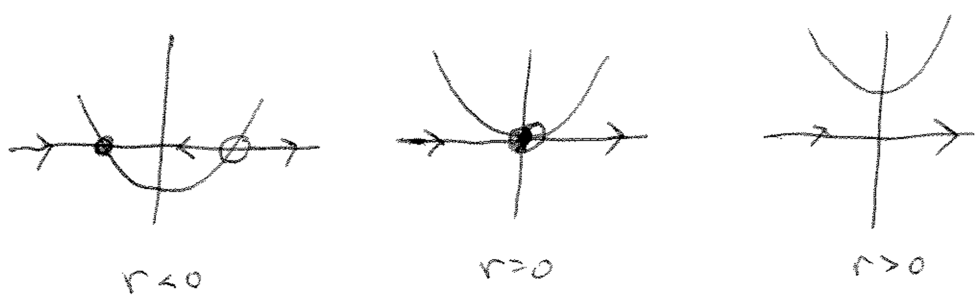


Fig 5.1 Phase planes for saddle-node bifurcation

For $r<0$, there are two stationary points at $x = \pm\sqrt{-r}$. The point on the left is stable that on the right is unstable. For $r=0$, there is a single stationary point, with a stable direction and an unstable direction. For $r>0$, there are no stationary points.

The bifurcation diagram is a plot of the stationary points as a function of r

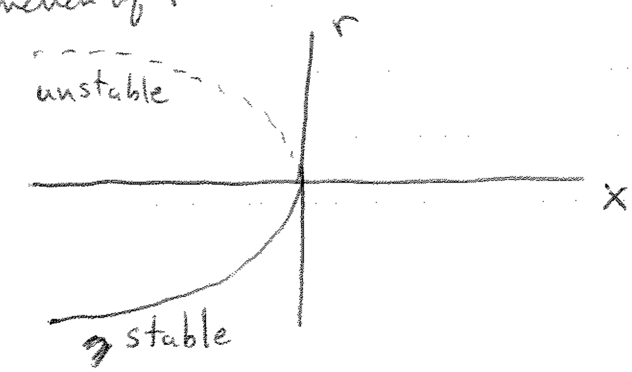


Fig. 5.2 Saddle-node bifurcation diagram

This bifurcation is also called a "fold" or a "turning point".

This bifurcation is stable, in the sense that any smooth, small perturbation of $f(x,r) = r + x^2$, will lead to a bifurcation of the same type. To see this let

$$g(x,r) = f(x,r) + \varepsilon f_1(x,r) \qquad (5.3)$$

For $r=0$, f has a double root at $x=0$, i.e.

$$f(0,0) = f_x(0,0) = 0 \qquad f_{xx}(0,0) \neq 0 \qquad (5.4)$$

Moreover for $r>0$, f has no roots near $x=0$ and for $r<0$ f has 2 distinct simple roots

$$\begin{aligned} f(\bar{x}_1, r) &= 0 \qquad f_x(\bar{x}_1, r) \neq 0 \\ f(\bar{x}_2, r) &= 0 \qquad f_x(\bar{x}_2, r) \neq 0 \end{aligned} \qquad (5.5)$$

For g, one can show the same behavior. There is a value $r = r_0$, for which g has a double root at $x = x_0$. For $r > r_0$, g has no roots near (x_0, r_0). For $r < r_0$, g has two simple roots near (x_0, r_0).

Consider a neighborhood of $(x, r) = (0,0)$. If ε is small, then $g_{xx} > 0$ and $g > 0$ for x away from 0. It follows that g has either two simple roots, one double root or no roots in x for each value of r.

For $r < 0$, f has two simple roots and $f_x < 0$. It follows that for ε small, $g_x < 0$ and one can solve $g(x, r) = 0$ for x near $\bar{x}_1$ and for x near $\bar{x}_2$.

For $r > 0$, f has no roots and $f_x > 0$. It follows that for ε small, $g_x > 0$ and g has no roots.

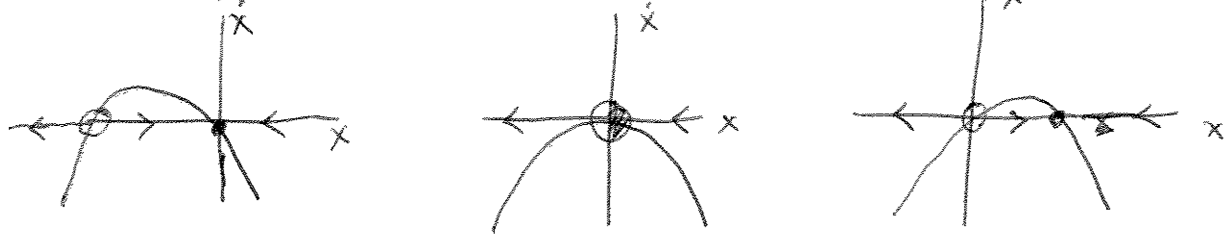
Let r_0 be the sup of the values of r with two simple roots. It follows that there is a double root of g at r_0. Because $f_r > 0$, then $g_r > 0$, from which it follows that g has no roots for $r > r_0$.

Finally the function $f(x, r) = r + x^2$ is a "normal form" for the saddle-node bifurcation. For any ODE having a saddle-node bifurcation, the flow is nearly given by $x' = f(x, r)$ in a neighborhood of the bifurcation.

<u>Transcritical bifurcation</u> Consider the equation

$$\dot{r} = rx - x^2 \tag{5.6}$$

This has a critical point fixed at $x=0$ for all r. This can occur due to the special nature of the value of $x=0$, for example if x = population.



$r<0$ $r=0$ $r>0$

Fig 5.3 Phase plane for transcritical bifurcation.

This bifurcation diagram is

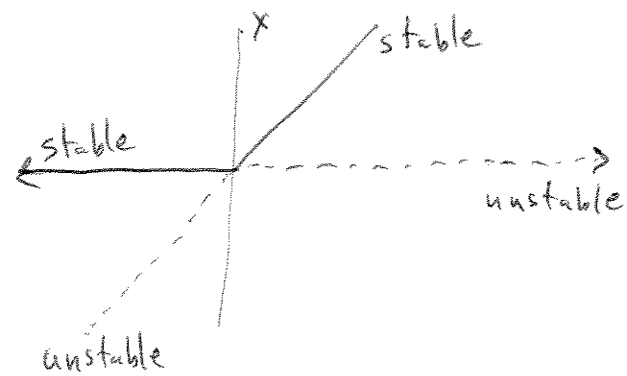


Fig. 5.4 Transcritical Bifurcation diagram

Notice that there is an exchange of stabilities as r crosses 0.

Pitchfork bifurcation

The symmetries of a problem may dictate that stationary points are created or destroyed in pairs. This gives rise to pitchfork bifurcations

Supercritical pitchfork bifurcation. Consider

$$\dot{x} = rx - x^3 \tag{5.7}$$

This has the following phase planes

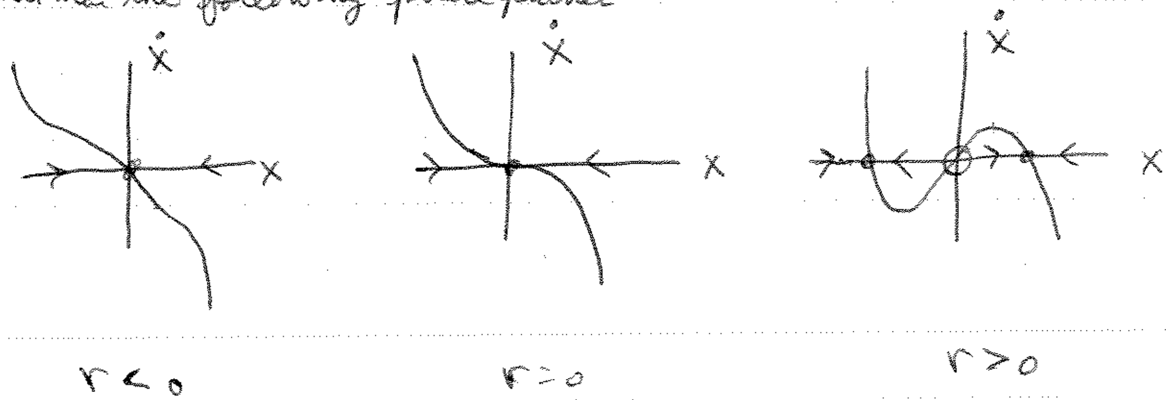


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Fig. 5.5 Phase planes for supercritical pitchfork bifurcation.

The bifurcation diagram ~~at the~~ shows a creation of ~~2~~ stable branches of stationary points and a transition from stable to unstable for the other stationary point as r crosses 0.

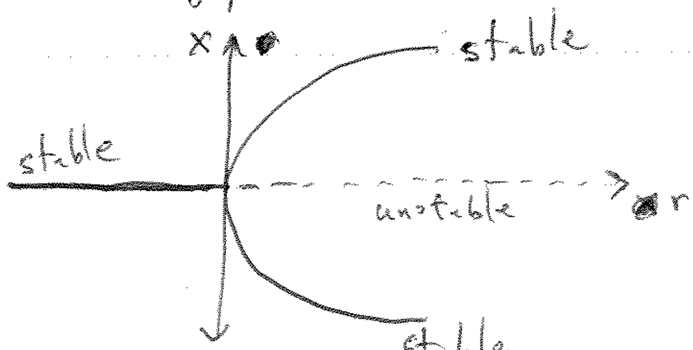


Fig. 5.6 Supercritical pitchfork bifurcation diagram

<u>Subcritical pitchfork bifurcation</u> This has the normal form

$$\dot{x} = rx + x^3 \tag{5.8}$$

with phase planes and bifurcation diagram as shown

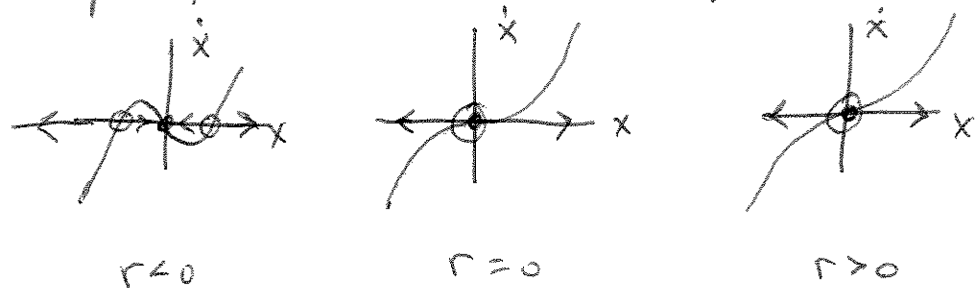
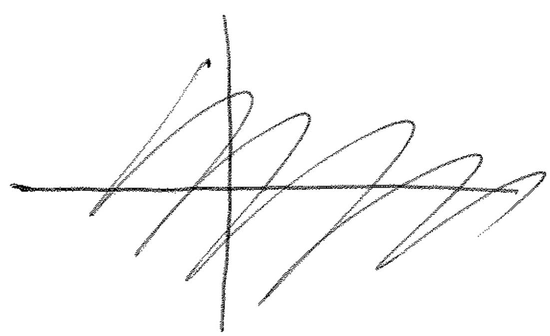


Fig 5.7 Phase planes for subcritical pitchfork bifurcation



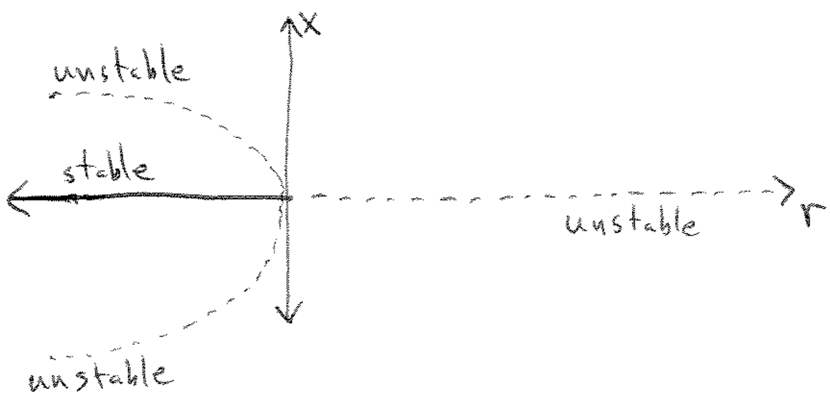


Fig. 5.8 Subcritical pitchfork bifurcation diagram

These bifurcation can be combined. A frequent example is

$$\dot{x} = rx + x^3 - x^5 \qquad (5.9)$$

This has the following diagram, which shows that it is a combination of a subcritical bifurcation and two saddle-node bifurcations

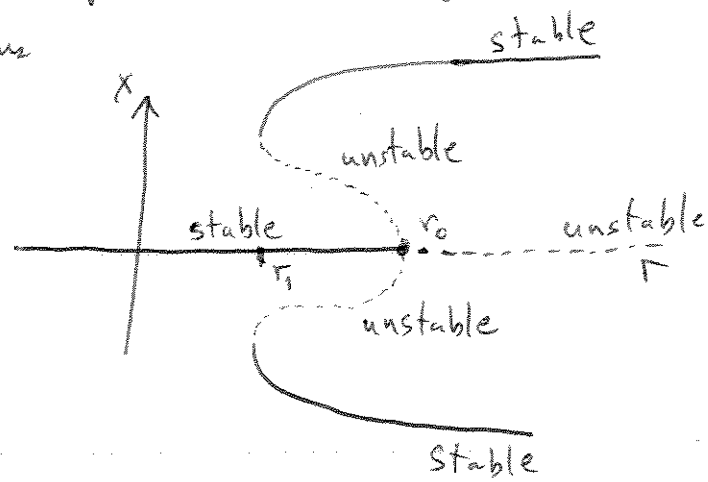


Fig. 5.9 ~~SM~~ Stabilized subcritical bifurcation diagram

This system can exhibit hysteresis. A solution on the stable fixed point $x=0$ for $r<r_0$, will go unstable as r crosses r_0. It may then jump to one of the stable branches. If r is then lowered it will move along that branch until r crosses r_1, where it will then jump down to $x=0$.

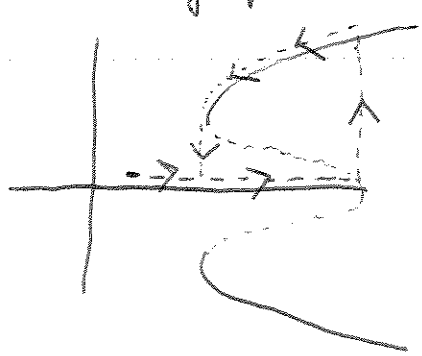


Fig. 5.10 Hysteresis loop for (5.9).

Note that for first order systems, only the saddle-node bifurcation is stable to all perturbations. The following figures show perturbations that break the transcritical and pitchfork bifurcations.

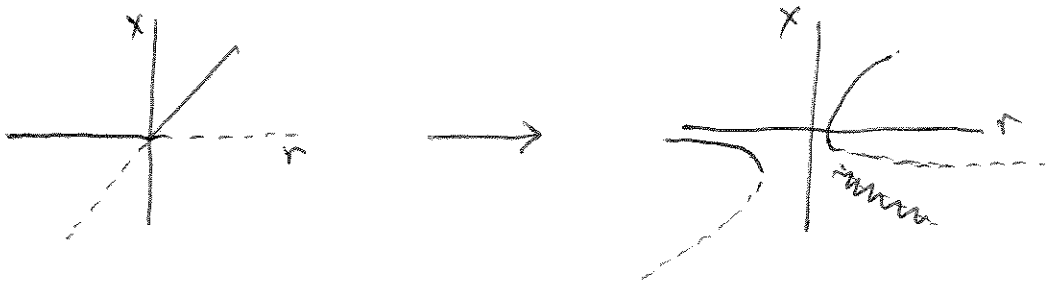


Fig 5.11 Perturbation of transcritical bifurcation into two saddle-node bifurcations

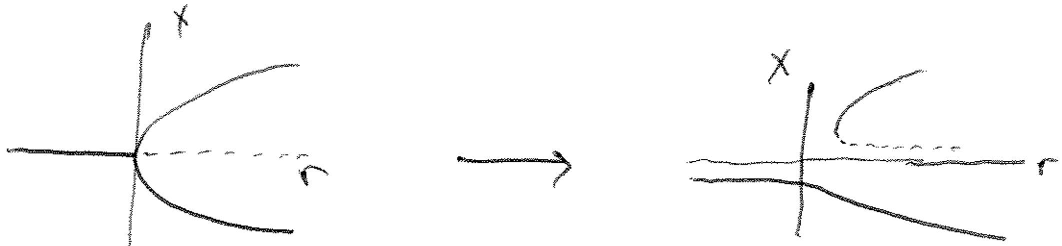


Fig 5.12 Perturbation of transcritical bifurcation into a saddle-node bifurcation and an unbifurcated branch.

These bifurcations are stable, however, in the presence of additional constraint, e.g. sysmmetry with respect to $x \to -x$.

5.2 Bifurcations for Second Order Systems

A simple extension of the bifurcations for 1st order makes them into bifurcations for 2nd order systems: Just add on an equation

$$\dot{y} = -y \qquad (5.10)$$

For example the saddle-node bifurcation for a 2nd order system has the normal form

$$\dot{x} = r + x^2 \qquad (5.11)$$
$$\dot{y} = -y$$

for which the phase plane plots are below.

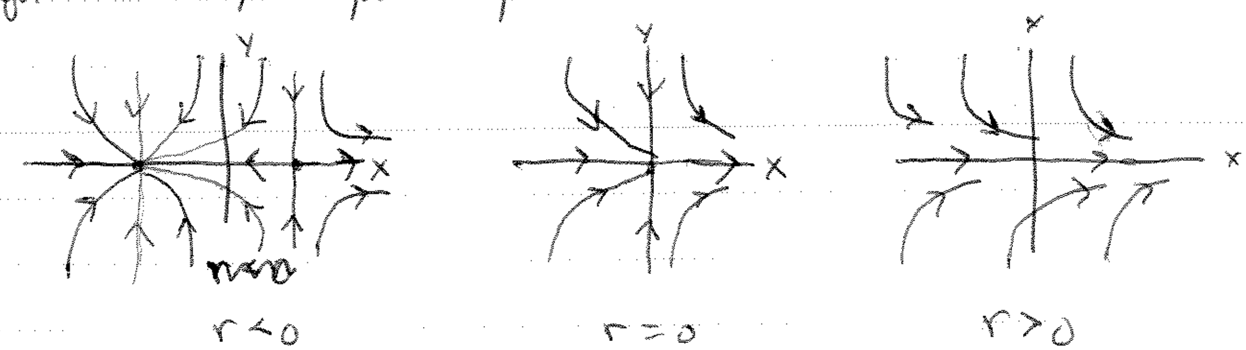


Fig. 5.13 Phase planes for Saddle-Node bifurcation in a second-order system

The bifurcations are still essentially one-dimensional in that the stationary points move towards each other on the one-dimensional "unstable manifold" of the saddle point.

This is also true of the other 3 bifurcations, given by

<u>transcritical</u> $\dot{x} = rx - x^2 \qquad \dot{y} = -y$ (5.12)

supercritical pitchfork: $\dot{x} = rx - x^3, \qquad \dot{y} = -y$ (5.13)

subcritical pitchfork: $\dot{x} = rx + x^3, \qquad \dot{y} = -y$ (5.14)

In each of these bifurcations there is a transition from a stable to an unstable fixed point. ~~This~~ For the stable fixed point (before the transition), both eigenvalues λ and μ for the linearized system are negative. After the transition at least one of the eigenvalues must be positive for the unstable fixed points(s). Let this e-value be λ. Then right at the transition point $\lambda = 0$.

The transitions above ~~involve~~ involve only changes in x that remain real. It is also possible for ~~x to go~~ the system to go from stable to unstable through development of complex values. This is called a Hopf bifurcation

In a supercritical Hopf bifurcation, exemplified by

$$\begin{aligned} \dot{r} &= \mu r - r^3 \\ \dot{\theta} &= \omega + br^2 \end{aligned} \qquad (5.15)$$

a critical point that has dying oscillations around it (i.e., a stable spiral) for $\mu < \mu_c = 0$ changes into a critical point with growing oscillations (an unstable spiral) for $\mu > \mu_c = 0$. In this example, the orbits go out to a limit cycle $r = \sqrt{\mu}$.

Fig 5.14 Phase planes for a Hopf bifurcation

Note that ~~for~~ near the critical point $\mu=0$, the size of the limit cycle $r=\sqrt{\mu}$ is small.

There is also a subcritical Hopf bifurcation. An example is

$$\dot r = \mu r + r^3 - r^5 \qquad (5.16)$$
$$\dot\theta = \omega + b r^2$$

In this example, there are ~~two~~ limit cycles, at r satisfying

$$\mu r + r^3 - r^5 = 0 \qquad (5.17)$$

with $r>0$.

$$-\mu - r^2 + r^4 = 0$$
$$r_\pm^2 = \tfrac{1}{2}\left(1 \pm \sqrt{1+4\mu}\right) \qquad (5.18)$$

There is also a critical point $r=0$. For $\mu<0$, $\dot r<0$ for r near 0, so that the critical point is a stable spiral. There are also two limit, since the formula in (5.18) is positive if $0>\mu>-\frac{1}{4}$.

If $\mu>0$, the critical point is an unstable spiral and there is a single limit cycle.

Fig. 5.15 Phase planes for a subcritical Hopf bifurcation.

In this subcritical case, the system, going through the bifurcation point, will change from have decaying oscillations to rapidly growing oscillations going to a limit cycle with radius $r = r_+ \cong 1$. So the solution jump from $r=0$ to $r=1$! This is one of the processes that can lead to chaos.