4. Methods for Reducing PDEs to ODEs

The source of ODEs is often PDEs and special solutions for them. Here a number of techniques for solving PDEs are described, all of which lead to ODEs.

4.1 Separation of Variables

This is one of the simplest and most widely used methods. It work for linear PDEs. As an example consider the Schrodinger equation

$$i\,\partial_t \psi = -\Delta\psi + V\psi \tag{4.1}$$

If the potential is

$$V = V(x,y,z) = V_1(x) + V_2(y) + V_3(z) \tag{4.2}$$

then look for

$$\psi = \psi(x,y,z,t) = e^{i\lambda t}\,\psi_1(x)\,\psi_2(y)\,\psi_3(z) \tag{4.3}$$

with

$$\lambda = \lambda_1 + \lambda_2 + \lambda_3 \tag{4.4}$$

~~Equation (4.1)~~

Substitute (4.2), (4.3), (4.4) in (4.1) to get

$$-(\lambda_1+\lambda_2+\lambda_3)\,\psi_1\psi_2\psi_3\,e^{i\lambda t} = -e^{i\lambda t}\left(\psi_{1xx}\psi_2\psi_3 + \psi_1\psi_{2yy}\psi_3 + \psi_1\psi_2\psi_{3zz}\right) + e^{i\lambda t}\left(V_1+V_2+V_3\right)\psi_1\psi_2\psi_3 \tag{4.5}$$

Divide by $\psi_1\psi_2\psi_3 e^{i\lambda t}$ to get

$$-(\lambda_1+\lambda_2+\lambda_3) = (-\psi_{1xx} + V_1\psi_1)/\psi_1 + (-\psi_{2yy} + V_2\psi_2)/\psi_2 + (-\psi_{3zz} + V_3\psi_3/\psi_3 \tag{4.6}$$

~~Ex~~ Since the only term depending on x is the first on the right, it must be constant. The same is true for the second and third ~~th~~ terms. Now we can identify the constants with these terms to get

$$-\psi_{1xx} + V_1\psi_1 = -\lambda_1\psi_1$$
$$-\psi_{2yy} + V_2\psi_2 = -\lambda_2\psi_2 \qquad (4.7)$$
$$-\psi_{3zz} + V_3\psi_3 = -\lambda_3\psi_3$$

If $V_1(x) = ax^2$, then there is an exact solution for the first equation. Otherwise a numerical solution is required.

4.2 Characteristics

Consider the nonlinear, inviscid Burgers equation with ~~space~~ forcing

$$u_t + uu_x = \alpha u \qquad (4.8)$$
$$u(x,0) = u_0(x)$$

~~Consider~~ Think of u as a velocity for a particle with position $X(t)$ i.e.

$$X_t(t) = u(X(t),t) \qquad (4.9)$$

Then

$$\frac{d}{dt}u(X(t),t) = u_x X_t + u_t$$
$$= u_t + uu_x$$
$$= \alpha u(X(t),t) \qquad (4.10)$$

The solution is

$$u(X(t), t) = u_0(x_0) e^{\alpha t} \qquad (4.11)$$

in which

$$\cancel{u_0 = u(x_0, 0)} \qquad (4.12)$$

$$x_0 = X(0)$$

The equation (4.9) for X is then

$$X_t = u_0 e^{\alpha t} \qquad (4.13)$$

$$X(0) = x_0$$

for which the solution is

$$X(t) = x_0 + \alpha^{-1} u_0 (e^{\alpha t} - 1) \qquad (4.14)$$

The two equations (4.11), (4.14) can be written as

$$\cancel{u(x,t) = u_0(x_0) e^{\alpha t}}$$

$$\cancel{x = x_0 + \alpha^{-1} u_0(x_0)}$$

$$u(x,t) = u_0(x_0)\, e^{\alpha t} \qquad (4.15)$$

$$x = x_0 + \alpha^{-1} u_0(x_0)\,(e^{\alpha t} - 1) \qquad (4.16)$$

One can invert (4.16) to get $x_0 = X_0(x,t)$, then insert this into (4.15) to get $u(x,t)$.

4.3 Traveling Wave Solutions

Traveling wave solution have the form

$$u(x,t) = u(x-st) \qquad (4.17)$$

This solution is constant on the line $x = x_0 + st$.

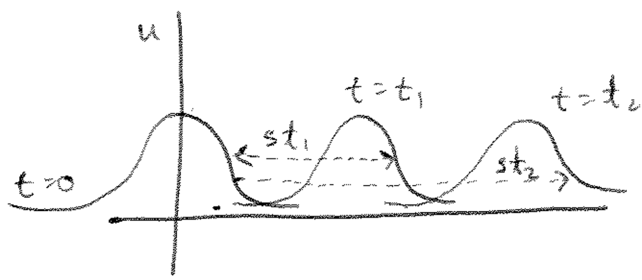

Fig 4.1

Consider the viscous Burgers equation

$$u_t + u u_x = \varepsilon u_{xx} \qquad (4.18)$$

Insert (4.17) to get, using $u_t = -s u_x$

$$-s u_x + u u_x = \varepsilon u_{xx} \qquad (4.19)$$

Since the left side is $(-su + \frac{1}{2}u^2)_x$, then integrate once to get

$$C - su + \tfrac{1}{2}u^2 = \varepsilon u_x \qquad (4.20)$$

Let

$$u_\pm = \lim_{x \to \pm\infty} u(x) \qquad (4.21)$$

the left side must be 0 at $u = u_+$, and $u = u_-$, so that

$$C - su + \tfrac{1}{2}u^2 = \tfrac{1}{2}(u - u_+)(u - u_-) \qquad (4.22)$$

i.e.

$$s = \tfrac{1}{2}(u_+ + u_-) \qquad (4.23)$$
$$c = \tfrac{1}{2}u_+ u_-$$

The solution of

$$\varepsilon u_x = \tfrac{1}{2}(u-u_-)(u-u_+) \qquad (4.24)$$

is

$$u = \frac{u_+ + u_-}{2} + \frac{u_+ - u_-}{2}\tanh\big(a(x-x_0)\big) \qquad (4.25)$$

in which a is a constant depending on ε and $(u_+ - u_-)/2$ and x_0 is an arbitrary shift.

4.4 Similarity Solutions

Another special form for a solution is

$$u(x,t) = t^{-\alpha}\, U(z = x/t^{\beta}) \qquad (4.26)$$

This has a solution where values and spatial dependence vary together without changing the shape of u

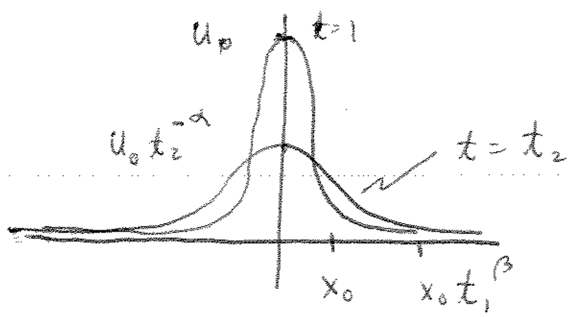

Fig 4.2

Consider the nonlinear heat equation

$$u_t - u_{xx} = u^2 \qquad (4.27)$$

Insert (4.20) into (4.21) to obtain

$$-\alpha t^{-\alpha-1} U - \beta t^{-\alpha-\beta-1} x U_z = t^{-2\alpha} U^2 + t^{-\alpha-2\beta} U_{zz} \qquad (4.28)$$

~~Observe~~
The term $t^{-\alpha-\beta} x = t^{-\alpha-1} z$, Eliminate t by setting

$$\alpha+1 = 2\alpha = \alpha + 2\beta$$

i.e.

$$\alpha = 1, \quad \beta = 1/2 \qquad (4.29)$$

Divide by $t^{-\alpha-1}$ to get

$$-U - \tfrac{1}{2} z U_z - U_{zz} = U^2 \qquad (4.30)$$

This ODE can be solved numerically.

5. Particle dynamics

The ~~incompressible~~ Euler equations for an incompressible, inviscid fluid in 2D are

$$\underset{\sim}{u}_t + \underset{\sim}{u}\cdot\underset{\sim}{\nabla} u + \underset{\sim}{\nabla} p = 0 \qquad (4.31)$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 0 \qquad (4.32)$$

in which $\underset{\sim}{u}$ is velocity and p is pressure, with

$$\underset{\sim}{u} = (u, v) \qquad (4.33)$$

$$\underset{\sim}{\nabla} = (\partial_x, \partial_y)$$

Define the vorticity ω as

$$\omega = \nabla\times u = v_x - u_y . \qquad (4.34)$$

~~Apply $\nabla\times$ to (4.31)~~

The x and y components of (4.31) are

$$u_t + (u u_x + v u_y) + p_x = 0 \qquad (4.35a)$$

$$v_t + (u v_x + v v_y) + p_y = 0 \qquad (4.35b)$$

Apply curl $= \nabla\times$ to (4.31), i.e.,

$$\partial_x\left(v_t + (u v_x + v v_y) + p_y\right) - \partial_y\left(u_t + (u u_x + v u_y) + p_x\right) = 0 \qquad (4.37)$$

The p terms cancel. The ∂_t terms are $\partial_t\,\omega$. The remaining nonlinear terms are

$$
\begin{aligned}
&(u v_x + v v_y)_x - (u u_x + v u_y)_y \\
&\quad = \left(u(v_{xx} - u_{xy}) + v(v_{xy} - u_{yy})\right) \\
&\quad\quad + u_x v_x + v_x v_y - u_y u_x - v_y u_y \\
&\quad = (u\,\omega_x + v\,\omega_y) + (v_x - u_y)(u_x + v_y) \\
&\quad = u\,\omega_x + v\,\omega_y = \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\,\omega
\end{aligned} \qquad (4.38)
$$

So the equation for ω is

$$\omega_t + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\,\omega = 0 \qquad (4.39)$$

In addition, since $u_x + v_y = 0$, then there is a "stream function" ψ for which

$$u = -\psi_y \tag{4.40}$$

$$v = \psi_x$$

$$\omega = v_x - u_y = \psi_{xx} + \psi_{yy}. \tag{4.41}$$

Now form a particle representation for ω

$$\omega(\underset{\sim}{x}) = \sum_{k=1}^{N} \omega_k \, \delta(\underset{\sim}{x} - \underset{\sim}{x}_k) \tag{4.42}$$

in which $\underset{\sim}{x}_k = \underset{\sim}{x}_k(t)$. This will satisfy (4.41) if

$$\nabla^2 \psi = \sum_{k=1}^{N} \omega_k \, \delta(\underset{\sim}{x} - \underset{\sim}{x}_k) \tag{4.43}$$

which has solution

$$\psi = \frac{1}{2\pi} \sum_{k=1}^{N} \log |\underset{\sim}{x} - \underset{\sim}{x}_k| \tag{4.44}$$

$$u = -\frac{1}{2\pi} \sum_{k=1}^{N} \frac{y - y_k}{|\underset{\sim}{x} - \underset{\sim}{x}_k|^2}$$

$$v = \frac{1}{2\pi} \sum_{k=1}^{N} \frac{x - x_k}{|\underset{\sim}{x} - \underset{\sim}{x}_k|^2} \tag{4.45}$$

The particles, called "point vortices" move at speed $\underset{\sim}{u}$, according to (4.39), i.e. $\dot{\underset{\sim}{x}}_k(t) = \underset{\sim}{u}(\underset{\sim}{x}_k(t), t)$. In the sum the contribution from $\underset{\sim}{x}_k$ itself is infinite, but it is an infinite spin which does not move the point, so we omit it to get

$$\frac{d}{dt}(x_k, y_k) = \frac{1}{2\pi} \sum_{j \neq k} \frac{(y_j - y_k, -(x_j - x_k))}{|\underset{\sim}{x}_j - x_k|^2} \tag{4.46}$$

This set of ODE's describes the motion of a collection of point vortices in a 2D incompressible, inviscid flow.