Now we construct the global properties of the orbits in the phase plane for the pendulum equation. First, the direction of the flow can be found.

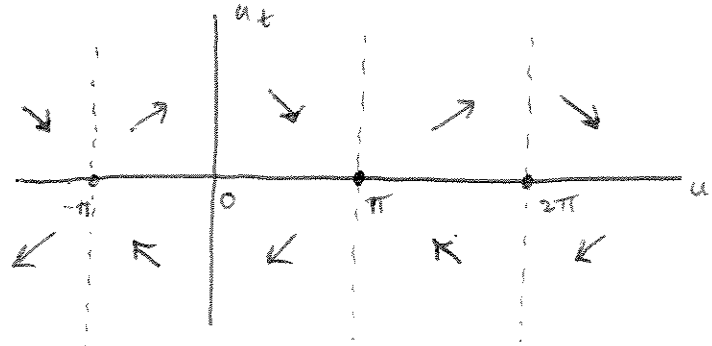


Fig. 3.15

The horizontal "velocity" u_t is to the right above the u-axis ($u_t > 0$) and to the left below the u-axis. ~~Thus~~ ($u_t < 0$). The vertical velocity $u_{tt} = -\sin u$ is up for $-\pi < u < 0$ and for $\pi < u < 0$. It is down for $0 < u < \pi$.

Next consider the symmetries of the flow. There is a symmetry around each of the lines $u_t = 0$, and $u = n\pi$. This can be seen directly from the velocity, but it is easier to see this for the energy $E = \frac{1}{2} u_t^2 - \cos u$; i.e.,

$$
\begin{aligned}
E(u, -u_t) &= E(u, u_t) \\
E(n\pi + \delta, -u_t) &= E(n\pi - \delta, u_t)
\end{aligned}
\qquad (3.46)
$$

Since the orbits are level curves of E, this implies that the ~~orbits~~ orbits are symmetric around ~~both~~ the lines $u_t = 0$ and $u = n\pi$.

Now we analyze the flow around the center $(0,0)$. Start at a point $(u_0, u_t=0)$ with $-\pi < u_0 < 0$. The orbit leaving this point goes up and to the right. It must eventually hit $u=0$, then it starts going down and to the right. Since the orbits are symmetric about $u=0$, this orbit will hit $u_t=0$ at a point $u_1 = -u_0$ with $0 < u_1 < \pi$. Since the flow is symmetric around $u_t = 0$, this orbit must continue in the lower half plane $u_t < 0$ until it comes back to u_0.

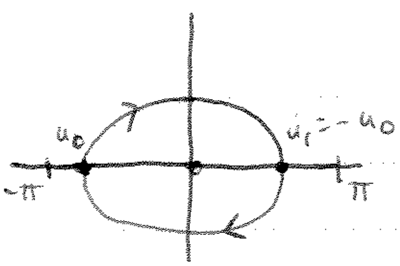


Fig. 3.16

The energy $E = \frac{1}{2}u_t^2 - \cos u$ varies from $E(0,0) = -1$ at $(0,0)$ to $E = 1$ as $u_1 \to \pi$. On the other hand, for points on the line $u=\pi$, $E > 1$. Since $E=1$ is the maximum value of E on $u_t = 0$, the orbits starting at $u = \pi$, $|u_t| > 0$ can never go through $u_t = 0$. These orbits go up and down as described in the figure 3.15. So they look like this

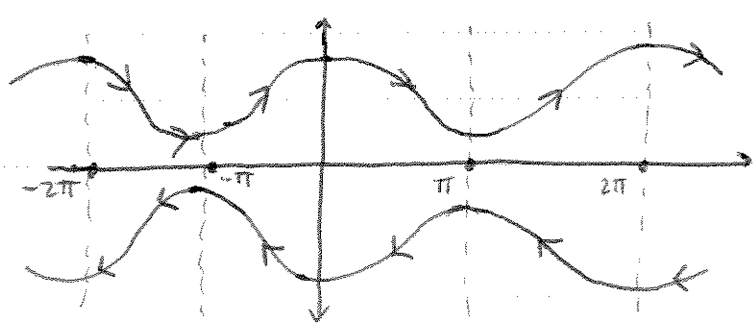


Fig. 3.17

In between the periodic orbits of Fig 3.16 and the extended orbits of Fig. 3.17, there are boundary curves. These are orbits that connect the saddle points $(-\pi, 0)$ and $(\pi, 0)$, as shown in Fig 3.18. These are called heteroclinic orbits.

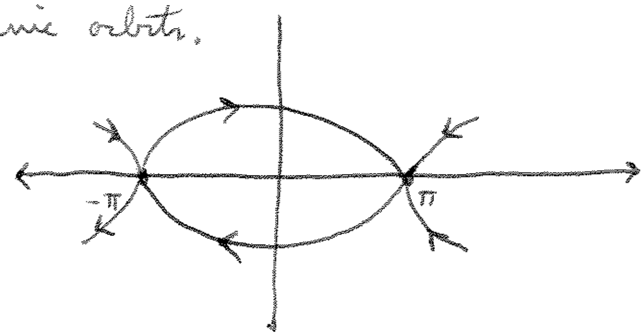


Fig 3.18

Putting this all together, we get the full phase plane for the pendulum equation in Fig. 3.19.

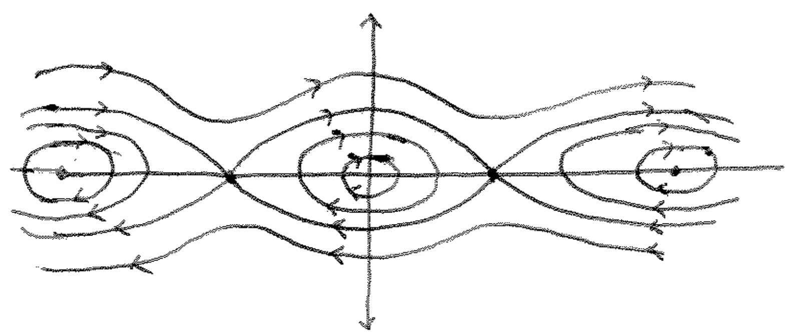


Fig 3.19

Poincaré - Bendixson Theorem

One property that make the 2D phase plane special is that ~~...~~ a closed curve ~~...~~ separates 2D into two disjoint regions. This isn't true in 3D. This property of 2D is called the Jordan curve theorem.

A consequence for ODEs is the Poincaré Bendixson Theorem.

Thm Consider the autonomous ODE $U' = F(U)$ with U a 2-vector. Suppose that Ω is a region for which the flow is entering Ω everywhere on $\partial\Omega$. The Ω contains either a fixed point or a periodic orbit.

We give an informal proof. The textbook contains a complete proof. First, ~~...~~ consider Ω that is simply connected. This means that every two points in Ω can be connected by a curve within Ω and the same is true for Ω^c. Then fixed point theorem of topology then shows that the flow has a stationary point.

Second suppose that Ω is not simply connected, with a single hole. Draw a curve C from the outer boundary of Ω to the inner boundary.

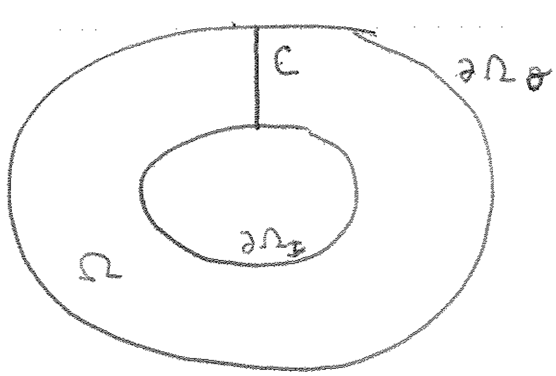


Fig. 3.20

Let U_0 be a point on C. One can show that the orbit ~~starting~~ starting from U_0 must hit C again at a point U_1. Denote $U_1 = PU_0$. The map $P: C \to C$ is called the Poincaré map. Since the flow is ~~exiting~~ entering on both $\partial\Omega_1$ and $\partial\Omega_2$, then if U_0 is close to the outer boundary $\partial\Omega_O$, then PU_0 is further from $\partial\Omega_O$ than U_0. Also if U_0' is close to $\partial\Omega_I$, then PU_0' is further from $\partial\Omega_I$.

Fig. 3.21

~~Repeated application of P~~

In addition if U_0 is to the left of U_0' on C, then PU_0 is to the left of PU_0' on C. ~~Then~~

Now apply P repeatedly to U_0 ~~until~~ the limit

$$U_\infty = \lim_{n\to\infty} P^n U_0 \qquad (3.47)$$

must stay to the left on U_0'. So it must be a point in the interior of Ω. Since $PU_\infty = U_\infty$, then this is a periodic orbit.

Nonlinearities at Stationary Points

Nonlinearities are significant for the stability of any marginally stable point.

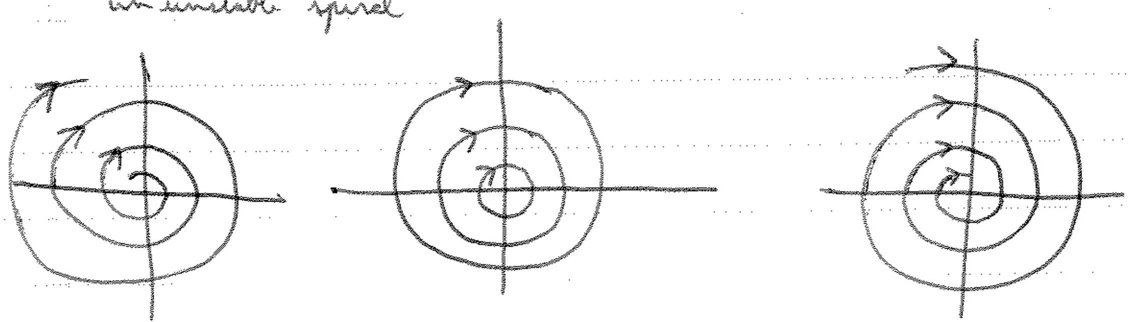
The first example is a center for the linearized flow. Consider the nonlinear perturbation of the harmonic oscillator

$$u_{tt} = -u_{\cancel{t}} - \varepsilon u_t^3 \qquad (3.48)$$

Denote $E = \frac{1}{2}(u_t^2 + u^2)$. Then

$$\begin{aligned} \frac{d}{dt} E &= u_t (u_{tt} + u) \\ &= -\varepsilon u_t^4 \\ &< 0 \quad \text{if } \varepsilon > 0 \end{aligned} \qquad (3.49)$$

if $u_t \neq 0$. This shows that the orbits are stable for (3.48). The phase plane is that of stable spiral. If $\varepsilon < 0$ it is an unstable spiral



$\varepsilon < 0$ $\qquad$ $\varepsilon = 0$ $\qquad$ $\varepsilon > 0$

Fig 3.22

As the next example consider any flow given by a phase plane portrait that differs from one of the ~~de~~ nondegenerate types. This can always be written as an ODE

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix}$$

Here's an example of a nonlinear saddle.

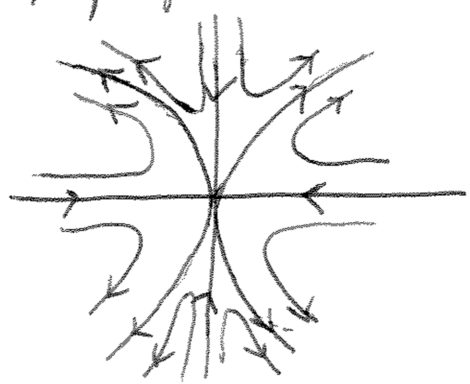


Fig. 3.23