MWF 10–10:50
Tues recitation
Th makeup

Math 266A
Fall 2006

I. Intro

ODE's occur throughout analysis and applied math

Examples of ODE's

(1) $y' = a(t)\,y + b(t) \qquad y(0) = y_0$

$$y = y_0 \exp\left(\int_0^t a(s)\,ds\right) + \int_0^t b(s)\, e^{\int_s^t a(s')\,ds'}\,ds$$

existence uniqueness (for all time) — first topic

(2) $y' = y^2 \qquad y(0) = y_0$

$\left(\frac{1}{y}\right)' = -\frac{y'}{y^2} = -1 \qquad \frac{1}{y} = \frac{1}{y_0} - t$

$$y = \frac{1}{y_0^{-1} - t}$$

blow up in finite time

(3) $y' = 2\sqrt{y} \qquad y = \begin{cases} 0 & t \le T \\ (t-T)^2 & T \le t \end{cases}$

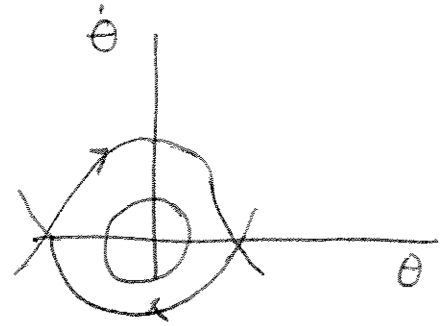
lack of uniqueness

(4) IVP for pendulum

$$\ddot{\theta} = -\frac{g}{L}\sin\theta$$

$$\theta(t=0) = \theta_0$$

$$\dot{\theta}(t=0) = \theta_1$$



phase plane method

(5) Heat distribution in rod

$$T_{xx} - \lambda T = S(x) \qquad \lambda: \text{removal rate}$$

$$T(0) = T(1) = 0 \qquad S = \text{source}$$

Expansion method - Sturm - Liouville Theory

Other topics

- linear systems of ODEs
- bifurcation + chaos
- linear algebra
- Fourier analysis

II. Initial Value Problems

<u>Thm (local existence)</u> Consider the ODE ~~[illegible]~~

$$y' = f(y,t) \qquad (2.1)$$
$$y(0) = y_0$$

Assume that

(i) f is defined in an open domain $D = \{ |t| < a, |y - y_0| < b \}$

(ii) f is uniformly Lipschitz in y in D; i.e.,

$$|f(y,t) - f(z,t)| < L|y-z| \qquad (2.2)$$

for all $(y,t) \in D$ and $(z,t) \in D$.

(iii) f is continuous in t in D (uniformly)

Then the ODE (2.1) has a unique solution $y(t)$ for some time interval $|t| < t_0$.

<u>Proof</u>

Rewrite the ODE in the equivalent integral form

$$y(t) = y_0 + \int_0^t f(y(s), s)\, ds \qquad (2.3)$$

Define the iteration

$$y^0(t) = 0 \qquad (2.4)$$
$$y^{n+1}(t) = y_0 + \int_0^t f(y^n(s), s)\, ds$$

The proof is in several steps

(i) Show that the iteration is well-defined; i.e.

$$(y^n(t), t) \in D \qquad (2.5)$$

for all n and for some time interval.

Let

$$M = \max_{(y,t) \in D} |f(y,t)| \qquad (2.6)$$

To show this, we first have $(y^0(t) = y_0,\ t) \in D$ for $|t| < a$.

Second ~~then~~ if $(y^n(t), t) \in D$ for $|t| < \alpha$, then

$$|y^{n+1} - y_0| \le \int_0^t |f(y^n(s), s)|\, ds \le tM \le \alpha M < b \qquad (2.7)$$

~~if~~ if $\alpha < (\frac{b}{M}, a)$. This shows that $(y^{n+1}(t), t) \in D$ for $|t| < \alpha$.

By iteration $(y^n(t), t) \in D$ for all n if $|t| < \alpha$.

(ii) Show that the iteration is convergent

Write

$$y^n = y^0 + \sum_{k=0}^{n-1} y^{k+1} - y^k \qquad (2.8)$$

Estimate

$$|y^{k+1}_{(t)} - y^k_{(t)}| = \left| \int_0^t f(y^k(s), s) - f(y^{k-1}(s), s)\, ds \right|$$

$$\le \int_0^t |f(y^k) - f(y^{k-1})|\, ds$$

$$\le \int_0^t L \max_{0 \le s \le t} |y^k - y^{k-1}|$$

$$\le tL \max |y^k - y^{k-1}| \qquad (2.9)$$

So

$$\max |y^{k+1} - y^k| \le tL \max |y^k - y^{k-1}| \qquad (2.10)$$

~~It follows~~
By applying this estimate repeatedly we get

$$
\begin{aligned}
\max |y^{k+1} - y^{k}| &\le tL \max |y^{k} - y^{k-1}| \\
&\le (tL)^2 \max |y^{k-1} - y^{k-2}| \\
&\cdots \\
&\le (tL)^k \max |y^{1} - y^{0}| \\
&\le (tL)^k \, (tM) \qquad (2.11)
\end{aligned}
$$

$= \gamma$

Choose $|t| < t_0$ with $t_0 L < 1$, then

$$(tL)^k < \gamma^k \qquad (2.12)$$

This shows that the terms in the series (2.8) are decreasing ~~th~~ algebraically. It follows that

$$y^n \xrightarrow[n]{} y = y_0 + \sum_{k=0}^{\infty} y^{k+1} - y^{k}$$

which is ~~uf~~ absolutely and uniformly convergent.

(iii) Show that the limit $y(t)$ is a solution.

We have $y^n \to y$ uniformly. It follows that y is continuous. Also the integral equation (2.7) has the following limit

$$
\begin{array}{ccccc}
y^{n+1}(t) & = & y_0 & + \int_0^t & f(y^n(s), s)\, ds \\
\downarrow & & \downarrow & & \downarrow \\
y(t) & = & y_0 & + \int_0^t & f(y(s), s)\, ds
\end{array}
\qquad (2.13)
$$

i.e., y solves equation (2.3).

(iv) Show uniqueness.

First prove the Gronwall inequality

<u>Gronwall inequality</u> If

$$x(t) \leq a\int_0^t x(s)ds + x_0 \qquad (2.14)$$

then

$$x(t) \leq x_0 e^{at} \qquad (2.15)$$

<u>Proof</u> Define

$$X(t) = a\int_0^t x(s)ds \qquad (2.16)$$

Then

$$x(t) \leq X(t) + x_0 \qquad (2.17)$$

So

$$X'(t) = a x(t) \leq a\left(X(t) + x_0\right) \qquad (2.18)$$

and after multiplying by e^{-at}

$$e^{-at}\left(X' - aX\right) \leq e^{-at} a x_0$$

i.e.

$$\frac{d}{dt}\left(e^{-at}X(t)\right) \leq e^{-at} a x_0$$

Since $X(0)=0$

$$e^{-at}X(t) \leq \int_0^t e^{-at'} a x_0 \, dt'$$

$$= \left(1 - e^{-at}\right) x_0$$

Multiply by e^{at} to get

$$X(t) \leq \left(e^{at} - 1\right) x_0$$

so that

$$x(t) \leq X(t) + x_0 \leq x_0 e^{at} \qquad (2.19)$$

Now suppose that y and z are two solutions of (2.3). Subtract to get

$$|y(t)-z(t)| = \left| \int_0^t f(y(s),s) - f(z(s),s)\, ds \right|$$

$$\leq L \int_0^t |y(s)-z(s)|\, ds \qquad (2.20)$$

$$X(0)-Z(0) = 0 \qquad (2.21)$$

Set $x(t) = y(t)-z(t)$. Then x satisfies the assumption of Gronwall's inequality with $a = L$ and $x_0 = 0$. Since $x_0 = 0$, then

$$X(t) = 0 \qquad (2.22)$$

i.e.

$$y = z \qquad (2.23)$$

This shows uniqueness

__Well-posedness.__ A differential equation, such as (2.1) is said to be __well-posed__ if the following hold

(i) existence

(ii) uniqueness

(iii) smooth dependence on parameters.

The system is well-posed for a time interval $|t| < t_0$ if (i), (ii) and (iii) hold in this time interval.

The ~~third~~ condition (iii) is often omitted from the definition of well-posedness. To explain the meaning of (iii) consider an ODE with a parameter

$$
\begin{aligned}
x_t(t,\lambda) &= f(x(t,\lambda), t, \lambda) \\
x(0,\lambda) &= x_0(\lambda)
\end{aligned}
\tag{2.24}
$$

We wish to show that x depends smoothly on λ.

As an example, set $f = f(x,t)$ and $x_0(\lambda) = \lambda$. Then the result is that the solution depends smoothly on the initial data.

__Thm__ Suppose that $f \in C^1(x,\lambda) \cap C^0(t)$ and that $x_0 \in C^1(\lambda)$. Then $x \in C^1(\lambda, t)$.

__Proof__ For each λ, the solution exists by the previous existence theorem. Take the derivative w.r.t. λ of (2.1) to get

$$
\begin{aligned}
\frac{d}{dt} x_\lambda &= f_x x_\lambda + f_\lambda \\
x_\lambda(0) &= x_{0\lambda}
\end{aligned}
\tag{2.25}
$$

For each λ, this is a linear equation for $y = x_\lambda$

$$\frac{d}{dt} y = g_1(t)\, y + g_2(t) \qquad (2.26)$$

$$y(0) = y_0$$

with

$$g_1(t) = f_x(\, x(t,\lambda),\ t,\ \lambda)$$

$$g_2(t) = f_\lambda(\, x(t,\lambda),\ t,\ \lambda) \qquad (2.27)$$

$$y_0 = x_{0\lambda}(0, \lambda)$$

Existence follows from the existence theorem. To see that $y = \frac{\partial}{\partial \lambda} x(t,\lambda)$, repeat the Picard proof (i.e. iteration) taking the λ-derivative at each step.

Systems of ODE's

The Picard Theorem is formulated for a first-order ODE

$$x' = f(x,t)$$

"First order" means that only the first derivative x' appears in the equation, along with x itself. The function x can be a vector

$$x(t) = (x_1(t), \dots, x_n(t)) \qquad (2.28)$$

and then ~~the function~~ f is a vector-valued function of the vector x and the scalar t

$$f = (f_1, \dots, f_n).$$

The only change in the Picard theorem is that $|x|$ refers to the vector length, i.e.

$$|x| = \sqrt{x_1^2 + \dots + x_n^2} \qquad (2.29)$$

Actually any other norm would work just as well.

Most ODE's can be written as first-order systems, so that this ~~restriction~~ feature of the Picard system is not much of a restriction. As an example consider a particle of mass m and position $x(t)$ moving in a force field $F(x,t)$. Newton's equation is force = mass x acceleration, which can be written as a second-order ODE

$$m x'' = F(x,t)$$

This is put in first order form by defining the ~~xxx~~ velocity $v = x'$ and the vector

$$\underline{X} = \begin{pmatrix} x \\ v \end{pmatrix} \qquad (2.30)$$

The equation for x and v are

$$x' = v \qquad (2.31)$$
$$v' = x'' = \frac{1}{m}F(x,t)$$

so that

$$\underline{X}' = \begin{pmatrix} x' \\ v' \end{pmatrix} = \begin{pmatrix} v \\ \frac{1}{m}F(x,t) \end{pmatrix}$$

$$= f(\underline{X}, t) \qquad (2.32)$$

in which

$$f(x,v,t) = \begin{pmatrix} v \\ \frac{1}{m}F(x,t) \end{pmatrix} \qquad (2.33)$$

Linear Systems of First Order ODE's

Let $x(t)$ be an n-vector function of t and let $A(t)$ be an $n \times n$ matrix function of t. Consider the linear, homogeneous first order system

$$X' = AX \qquad (2.34)$$
$$X(0) = X_0$$

In order to describe the solution of (2.34), we need some definitions and results from linear algebra. ~~We always write~~ Unless stated otherwise the vectors here are column vectors.

1. Eigenvalues and eigenvectors. λ is an eigenvalue and v is an eigenvector for a matrix A if

$$Av = \lambda v \qquad (2.35)$$

in which A is an $n \times n$ matrix, v is an n-vector and λ is a scalar

2. Inverse matrix. A^{-1} is the inverse matrix for A if

$$A^{-1}A = AA^{-1} = I \qquad (2.36)$$

in which I is the identity matrix

3. Similarity transformation. If

$$B = C^{-1}AC \qquad (2.37)$$

then B is similar to A

4. Orthogonal matrix. A matrix U is orthogonal if

$$U^{+} = U^{-1} \qquad (2.38)$$

in which $(U^{+})_{ij} = U_{ji}$.

5. Self-adjoint matrix. A ~~matrix~~ matrix A with real components is self adjoint if $A^{+} = A$, i.e.

$$A_{ij} = A_{ji} \tag{2.39}$$

6. Diagonalization of a self-adjoint matrix. If A is self-adjoint then

$$A = U \Lambda U^{+} \tag{2.40}$$

in which U is orthogonal and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is the $n \times n$ matrix with the eigenvalues λ_k of A on its diagonal, i.e.

$$\Lambda_{kk} = \lambda_k$$

$$\Lambda_{kl} = 0 \quad \text{if } k \neq l \tag{2.41}$$

Moreover the columns of U are the corresponding eigenvectors, i.e.

$$U = (v_1 \ldots v_n) \tag{2.42}$$

7. Dot product. For two (real) vectors x and y both of length n the dot product is

$$x \cdot y = \sum_{k=1}^{n} x_k y_k \tag{2.43}$$

This is an example of an inner product

$$(x, y) = x \cdot y = x^{+} y \tag{2.44}$$

Two vectors x and y are said to be orthogonal if $x \cdot y = 0$. More generally, the angle θ between two vectors satisfies

$$\cos\theta = \frac{x \cdot y}{|x|\,|y|} \tag{2.45}$$

8. <u>Orthogonality of eigenvectors</u>. Let A be a self adjoint matrix with eigenvectors v and w and corresponding eigenvalues λ and μ; i.e.

$$Av = \lambda v$$
$$Aw = \mu w$$

with $\lambda \neq \mu$. Then v and w are orthogonal

<u>Proof</u>

$$\begin{aligned} v \cdot w &= v \cdot \left(\tfrac{1}{\mu} A w\right) \\ &= \mu^{-1} \; v^{+}(Aw) \\ &= \mu^{-1} \; (v^{+}A) \; w \\ &= \mu^{-1} \; (A^{+}v)^{+} w \\ &= \mu^{-1} \; (Av)^{+} w \\ &= \mu^{-1} \; (\lambda v)^{+} w \\ &= \mu^{-1}\lambda \; v \cdot w \end{aligned}$$

Since $\frac{\lambda}{\mu} \neq 1$, then $v \cdot w = 0$.