Induced universal graphs

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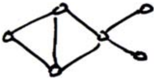
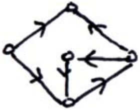
Types of graphs.

A graph consists of two sets, a vertex set V; and an edge set E which gives connections between vertices. Different assumptions about edges lead to different types of graphs.

Simple graphs: edges are unordered pairs of vertices (no repetitions)



Directed graphs: edges are ordered pairs of vertices

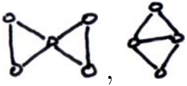


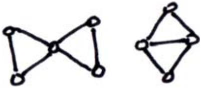
Multi-graphs: edges are unordered "pairs" of vertices (repetition allowed)

Universal graphs

Let $\mathcal{F}$ be a collection of graphs. Then we say that a graph U is a universal graph for $\mathcal{F}$ if every graph in $\mathcal{F}$ is a subgraph of U.

Example:

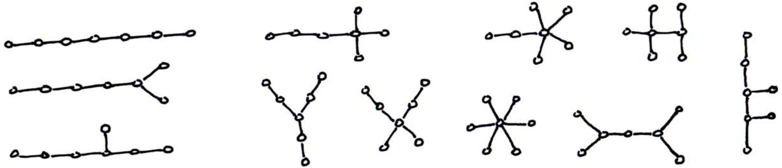
$\mathcal{F} = \{$  $\}$

$U_1 =$ 

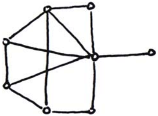
$U_2 =$ 

An example with trees

Let $\mathcal{F}$ be the set of trees on 7 vertices.



Then the following is a universal graph for $\mathcal{F}$.



Previously studied families for universal graphs

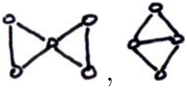
Families for which universal graphs have been studied include

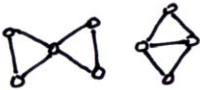
- Trees on n vertices [Bhatt et al. '89; Chung et al. '81; Friedman and Pipenger '87; Gol'dberg and Livšic '68; Nebeský '75; Yang '92]
- Planar graphs with bounded degree [Capalbo '02]
- Caterpillars [Chung and Graham '81]
- Cycles [Bondy '71]
- Sparse graphs [Babai et al. '82; Rodl '81]
- Graphs with bounded degree [Alon et al. and Capalbo et al. '99 '00 '01 '02 '07]

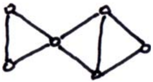
Induced universal graphs

Let $\mathcal{F}$ be a collection of graphs. Then we say that a graph U is an induced universal graph for $\mathcal{F}$ if every graph in $\mathcal{F}$ is an induced subgraph of U.

Example:

$\mathcal{F} = \{$  $\}$

$U_1 =$ 

$U_2 =$ 

An example with trees

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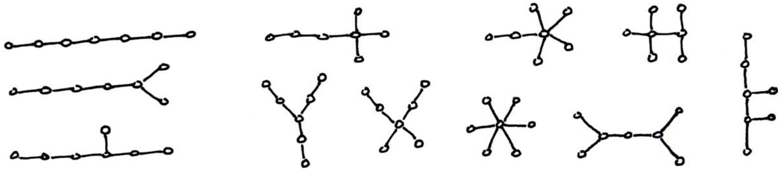
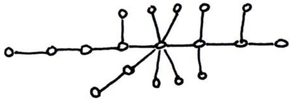


Then the following is an induced universal graph for $\mathcal{F}$.



Previously studied families for induced universal graphs

Families for which induced universal graphs have been studied include

- All graphs on *n* vertices [Moon ‘65]
- Tournaments [Moon ‘68]
- Trees on *n* vertices [Chung et al. ‘81]
- Planar graphs [Chung ‘90]
- Graphs with bounded arboricity [Chung ‘90]
- Graphs with bounded degree [Butler]

Main result

Main result

Let $\mathcal{F}$ be all graphs on n vertices with maximum degree at most r. Then there is an induced universal graph U such that

$$|V(U)| \leq Cn^{\lfloor (r+1)/2 \rfloor} \qquad \text{and} \qquad |E(U)| \leq Dn^{2\lfloor (r+1)/2 \rfloor - 1}$$

(C and D are constants depending only on r).

The proof consists of three major parts:

1. We can decompose our graphs into $\lfloor (r+1)/2 \rfloor$ graphs each one of which has degrees at most 2.
2. There is a small induced universal graph for the family of graphs on n vertices with maximum degree at most 2.
3. Small induced universal graphs can be combined to form large induced universal graphs.

Hall's Marriage Theorem

If there are m girls and any k of the girls likes (collectively) at least k of the boys (for $k = 1, 2, \ldots, m$), then it is possible for each girl to be paired with a boy she likes.

A graph is bipartite if we can split the vertices into disjoint sets V_1, V_2 so that all edges go between the sets. A perfect matching of the graph is a set of edges so that each vertex is incident to exactly one edge.

Perfect matchings in regular bipartite graphs

Every regular bipartite graph has a perfect matching. Moreover, the edges can be represented as a union of perfect matchings.

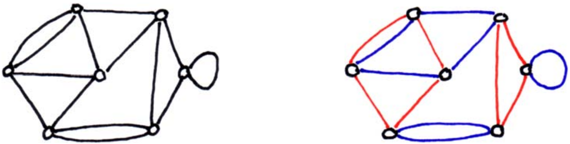
Proof: Let V_1 be the set of girls, V_2 the set of boys and edges indicate "liking", then we can use Hall's Marriage Theorem to find a pairing, which will correspond to a perfect matching.

Multigraph form of Petersen's Theorem

Multigraph form of Petersen's Theorem

If G is a multigraph on n vertices where the degree at each vertex is s (s even). Then G can be decomposed into $s/2$ edge disjoint graphs where the degree at each vertex is 2.

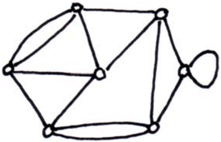
Example ($n = 7$, $s = 4$):

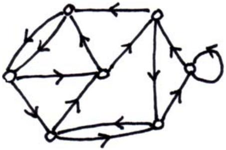


Proof or Petersen's Theorem (part I)

Without loss of generality assume the graph is connected. Since the degree at each vertex is even we can find an Eulerian cycle. Using the cycle orient the edges so that at each vertex the in-degree is equal to the out degree.

Example:

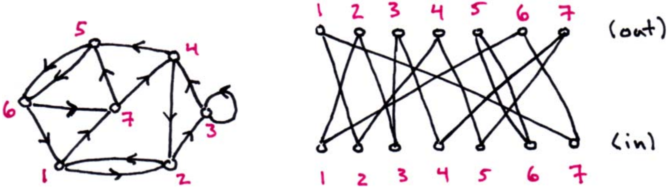




Proof or Petersen's Theorem (part II)

Split each vertex v into two denoted as v_{in} and v_{out}, this forms a regular bipartite graph. It follows that we can decompose the edges of this bipartite graph into a set of $s/2$ perfect matchings.

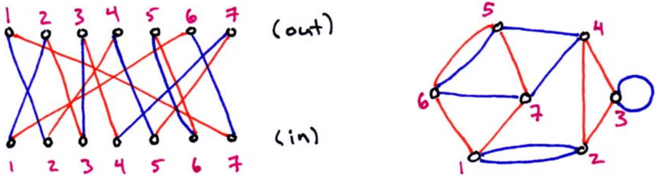
Example:



Proof or Petersen's Theorem (part III)

The set of edges of a perfect matching in the bipartite graph correspond to a subgraph in the original graph where the degree at each vertex is 2. This concludes the proof.

Example:



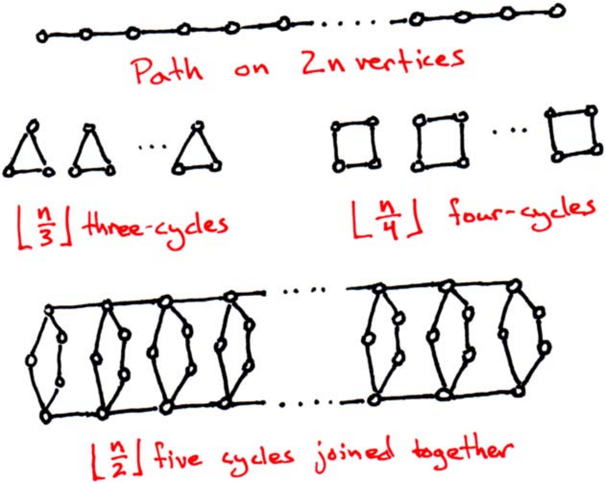
Decomposition result

Decomposition result

Let G be a graph on n vertices with maximum degree r. Then G can be decomposed into $\lfloor (r+1)/2 \rfloor$ edge disjoint subgraphs each with maximum degree at most 2.

Proof: Given G add enough edges so that G is r regular (if r is even) or $(r+1)$ regular (if r is odd). Then apply the previous result.

An induced universal graph U



Note: $|V(U)| \leq 6.5n$ and $|E(U)| \leq 7.5n$.

Checking that it is an induced universal graph

Lemma

The graph U shown above is an induced universal graph for the family of graphs on n vertices with maximum degree at most 2.

If G has maximum degree 2 then it is composed of paths and cycles. To show that the previous graph is an induced universal graph need to show how to embed any such G into the graph.

- Embed the paths of G in the long path of U.
- Embed the three-cycles of G in the three-cycles of U.
- Embed the four-cycles of G in the four-cycles of U.
- Embed the longer cycles of G in the long chain of five cycles in U.

How to embed longer cycles

Example:

To insert a cycle of length b we need to use $\lfloor b/2 \rfloor - 1$ five cycles. Since we also need to add a "buffer" five cycle between consecutive embedded cycles; it follows we need at most $\lfloor n/2 \rfloor$ five cycles strung together in order to embed all the cycles.

Combining induced universal graphs

Theorem (Chung '90)

Let $\mathcal{F}$ be a family of graphs and U a corresponding induced universal graph for $\mathcal{F}$. If $\mathcal{H}$ is a family of graphs where each graph can be broken into k subgraphs each belonging in $\mathcal{F}$, then there is an induced universal graph W for $\mathcal{H}$ such that

$$|V(W)| = |V(U)|^k \qquad \text{and} \qquad |E(W)| \leq k|V(U)|^{2k-2}|E(U)|.$$

Note: in general do not need to assume that all the $\mathcal{F}$ are the same, i.e., can have k different families of graphs and k different corresponding universal graphs.

Forming the larger induced universal graph

Given the graph U we form W as follows:

- Vertices of W are k-tuples of vertices of U, i.e., $(u_1, u_2, \ldots, u_k)$.
  - This gives W exactly $|V(U)|^k$ vertices.
- $(u_1, u_2, \ldots, u_k)$ is adjacenct to $(u'_1, u'_2, \ldots, u'_k)$ in W if and only if for some i the vertex u_i is adjacent to u'_i in U.
  - Any edge $\{u, u'\}$ in U can form at most $k|V(U)|^{2k-2}$ edges in W. Namely pick an i from 1 to k and then fix the two entries u_i and u'_i, finally all the remaining $2k-2$ entries can vary. So there are at most $k|V(U)|^{2k-2}|E(U)|$ edges.

Verifying that the graph works

Suppose that G is a graph in $\mathcal{H}$ and it can be broken into subgraphs $G_1, G_2, \ldots, G_k$ each of which is in $\mathcal{F}$. Since U is an induced universal graph for $\mathcal{F}$ each one of the G_i is an induced subgraph of U. For a vertex v of G_i (and hence of G) let $u_i(v)$ denote the vertex in U where it embeds. Now consider the following:

$$
\begin{aligned}
&(u_1(v), \ldots, u_k(v)) \text{ is adjacent to } (u_1(v'), \ldots, u_k(v')) \\
&\text{if and only if } u_i(v) \text{ is adjacent to } u_i(v') \text{ for some } i \\
&\text{if and only if } v \text{ is adjacent to } v' \text{ in } G_i \text{ for some } i \\
&\text{if and only if } v \text{ is adjacent to } v' \text{ in } G.
\end{aligned}
$$

So $v \mapsto (u_1(v), u_2(v), \ldots, u_k(v))$ embeds G as an induced subgraph of W.

Main result (revisited)

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Let $\mathcal{F}$ be all graphs on n vertices with maximum degree at most r. Then there is an induced universal graph U such that

$$|V(U)| \leq Cn^{\lfloor (r+1)/2 \rfloor} \qquad \text{and} \qquad |E(U)| \leq Dn^{2\lfloor (r+1)/2 \rfloor - 1}$$

(C and D are constants depending only on r).

The proof consists of three major parts:

1. We can decompose our graphs into $\lfloor (r+1)/2 \rfloor$ graphs each one of which has degrees at most 2.
2. There is a small induced universal graph for the family of graphs on n vertices with maximum degree at most 2.
3. Small induced universal graphs can be combined to form large induced universal graphs.

How good is the result?

We certainly have that the number of induced subgraphs of our induced universal graph is as least as large as the number of graphs in the family. So for n sufficiently large,

$$\frac{|V(U)|^n}{n!} \geq \binom{|V(U)|}{n} \geq |\mathcal{F}| \geq e^{-(r^2-1)/4}\left(\frac{r^{r/2}}{e^{r/2}r!}\right)^n n^{rn/2}/n!.$$

So $|V(U)| \geq cn^{r/2}$ for some constant c depending only on r.

So for r even within a constant multiple of smallest number of vertices. For r odd off by a factor of $n^{1/2}$.

We can generalize to other types of graphs

Multigraph result

Let $\mathcal{F}$ be all multi-graphs on n vertices with maximum degree at most r. Then there is an induced universal multi-graph U such that

$$|V(U)| \leq Cn^{\lfloor (r+1)/2 \rfloor} \qquad \text{and} \qquad |E(U)| \leq Dn^{2\lfloor (r+1)/2 \rfloor - 1}.$$

Directed graph result

Let $\mathcal{F}$ be all directed graphs on n vertices with maximum in-degree and out-degree at most r. Then there is an induced universal directed graph U such that

$$|V(U)| \leq Cn^{r} \qquad \text{and} \qquad |E(U)| \leq Dn^{2r-1}.$$

Concluding remarks

We have investigated the construction of small induced universal graphs for the family of graphs with bounded maximum degree.

- What about odd n?
  (Alon and Capalbo have done $n = 3$)
- How about other families of graphs?
  (Almost no results in the literature on directed graphs, multi-graphs or hypergraphs.)