Induced-universal graphs for graphs with bounded maximum degree

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Abstract. For a family $\mathcal{F}$ of graphs, a graph U is induced-universal for $\mathcal{F}$ if every graph in $\mathcal{F}$ is an induced subgraph of U. We give a construction for an induced-universal graph for the family of graphs on n vertices with degree at most r, which has $Cn^{\lfloor (r+1)/2 \rfloor}$ vertices and $Dn^{2\lfloor (r+1)/2 \rfloor - 1}$ edges, where C and D are constants depending only on r. This construction is nearly optimal when r is even in that such an induced-universal graph must have at least $cn^{r/2}$ vertices for some c depending only on r.

Our construction is explicit in that no probabilistic tools are needed to show that the graph exists or that a given graph is induced-universal. The construction also extends to multigraphs and directed graphs with bounded degree.

Key words. universal graphs, induced universal graphs, graph decomposition

1. Introduction

Except where indicated all graphs are assumed to be undirected and without loops or multiple edges. Our terminology is standard and any undefined terms can be found in standard graph theory books [17], [27].

For a family of graphs $\mathcal{F}$, a graph U is universal for $\mathcal{F}$ if every graph in $\mathcal{F}$ is a subgraph of U. The most frequently studied family for universal graphs has been the trees on n vertices [7], [13], [16], [18], [20], [21], [25], [28]. Other families that have been studied include planar graphs with bounded degrees [7], [10], caterpillars [15], cycles [8], and sparse graphs [6], [26]. After a series of improvements [1], [4], [5], [11], [10] Alon and Capalbo [2] were able to construct a universal graph for the family of all graphs on n vertices with bounded degree where the number of edges in the universal graph is within a small log factor of the fewest possible.

Related to universal graphs are the induced-universal graphs where for a family of graphs $\mathcal{F}$ a graph U is induced-universal if every graph in $\mathcal{F}$ is an induced subgraph of U. The family of all graphs on n vertices was considered by Moon [24], while Chung considered trees, planar graphs, and graphs with bounded arboricity [12] (also see [23]). In this paper we will focus on the family of all graphs on n vertices with bounded degree. Our main result is the following.

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Theorem 1. *Let $\mathcal{F}$ be the family of all graphs with maximum degree at most r on n vertices. Then there exists an induced-universal graph U for $\mathcal{F}$ where*

$$|V(U)| \leq Cn^{\lfloor (r+1)/2 \rfloor}, \quad \text{and} \quad |E(U)| \leq Dn^{2\lfloor (r+1)/2 \rfloor - 1},$$

where C and D are constants which depend only on r.

We will also show that the number of vertices is within a constant factor of the smallest possible number of vertices. We comment that our proof is constructive, in that we will give an explicit construction of an induced-universal graph and show how to embed any given graph in the family into the graph. This differs from the approach taken in [1], [2], [4], [5], [11], [10] for universal graphs which rely on probabilistic methods (either to show that a graph exists or to show that a given construction is a universal graph).

Our method will be to first find a decomposition of the graph into pieces with degree at most 2 (see Section 2). We will then form an induced-universal graph for graphs with degree at most 2 (see Section 3). By patching together these small induced-universal graphs (see Section 4) we will then form our desired induced-universal graph. The approach we give easily generalizes to multigraphs (graphs with loops and multiple edges), directed graphs, and directed multigraphs (see Section 5).

2. Decomposing our graphs

The first component of our proof is a multigraph version of Petersen's Theorem. We will not give the proof here but it can easily be adopted from standard proofs (see [17], [27]).

Theorem 2. (Petersen's 2-factorable Theorem) *Let G be a multigraph on n vertices where the degree at each vertex is s, with s even. Then G can be decomposed into $s/2$ edge disjoint graphs where the degree at each vertex is 2.*

This decomposition of regular multigraphs can be used to give a decomposition of graphs. Namely starting with a graph with maximum degree s we can add edges (possibly multiple edges or loops) until the degree at each vertex is s (if s is even) or $s+1$ (if s is odd). Now applying Theorem 2 we can decompose this new graph into at most $\lfloor (s+1)/2 \rfloor$ edge disjoint subgraphs each of which is regular of degree 2. Finally by removing any edges that we initially added we end up with a decomposition of our original graph into subgraphs with maximum degree at most 2. This establishes the following corollary.

Corollary 1. *Let G be a graph on n vertices with maximum degree s. Then G can be decomposed into $\lfloor (s+1)/2 \rfloor$ edge disjoint subgraphs where the maximum degree of each such subgraph is at most 2.*

3. A simple induced-universal graph

To take advantage of our decomposition result we need an induced-universal graph for the family $\mathcal{F}$ of graphs on n vertices with maximum degree at most 2. In this section we will show that the graph U shown in Figure 1 is such a graph.

Note for this graph we have

$$|V(U)| = 2n + 3\left\lfloor \frac{n}{3} \right\rfloor + 4\left\lfloor \frac{n}{4} \right\rfloor + 5\left\lfloor \frac{n}{2} \right\rfloor \leq 6.5n,$$
$$|E(U)| = (2n-1) + 3\left\lfloor \frac{n}{3} \right\rfloor + 4\left\lfloor \frac{n}{4} \right\rfloor + (7\left\lfloor \frac{n}{2} \right\rfloor - 2) \leq 7.5n.$$

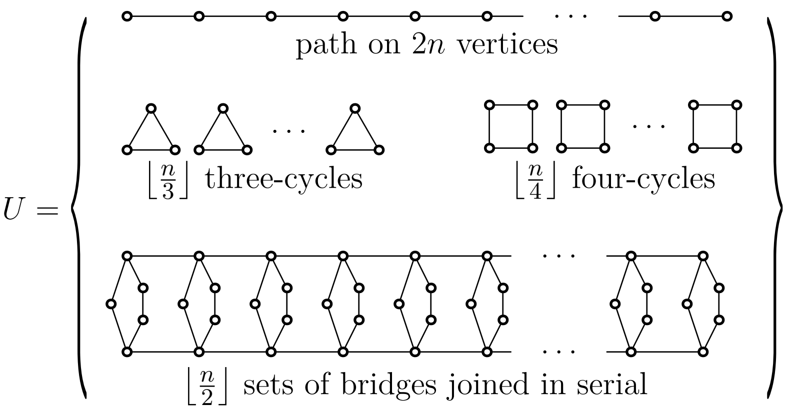

Fig. 1. An induced-universal graph for the family of graphs on n vertices with maximum degree at most 2.

We have not made any attempt to find the smallest possible U. For instance Esperet, Labourel and Ochem [19] have been able to modify this construction to get slightly fewer vertices and edges. For our purposes it is enough to see that we are within some constant factor of the smallest possible.

Lemma 1. *The graph U given in Figure 1 is an induced-universal graph for the family $\mathcal{F}$ of all graphs on n vertices with maximum degree at most* 2*.*

Proof. We need to show that each $G \in \mathcal{F}$ can be embedded in U, we first note that a graph $G \in \mathcal{F}$ is composed of cycles and paths (we consider isolated vertices as paths with one vertex).

Clearly we can embed the paths into the long path of U and any 3 or 4 cycles can be embedded. So now suppose that we have cycles of length $b_1, b_2, \ldots, b_q$ all of length at least 5. Then we will use the sets of bridges that have been connected in serial to embed these cycles as induced subgraphs. In the case that $b_q = 3 + 2k$ we use k of the sets of bridges by taking 2 edges in a bridge off the first set, then use $2(k-1)$ of the edges connecting the bridges, then finally using 3 more edges on a bridge on the last set. In the case $b_q = 2+2k$ then we again use k of the sets of bridges where we take 2 edges in a bridge of the first set, use $2(k-1)$ of the edges connecting the bridges, then finally using 2 more edges on a bridge of the last set. Examples are shown in Figure 2 for cycles of length 8 and 11.

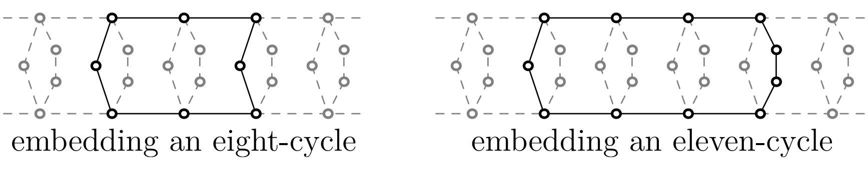

Fig. 2. Embedding long cycles into U.

Note that by this procedure we will use $\lfloor b_k/2 \rfloor - 1$ of the sets of bridges to insert a cycle of length b_k as an induced subgraph. We also will need to skip a set of bridges between induced cycles (to prevent any edges becoming induced that should not be). Therefore we need at most

$$\left\lfloor \frac{b_1}{2} \right\rfloor + \left\lfloor \frac{b_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{b_q}{2} \right\rfloor - 1 \ \leq\ \frac{1}{2}(b_1 + b_2 + \cdots + b_q) - 1 \leq \left\lfloor \frac{n}{2} \right\rfloor$$

of these sets of bridges in serial (hence our construction). □

4. Forming large induced-universal graphs

The final tool we need to prove Theorem 1 is a way to patch the induced-universal graphs together for the graphs in our decomposition to form a larger induced-universal graph. This will be done with the following construction due to Chung [12] which we include here to show how to embed graphs into the induced-universal graph.

Theorem 3. *Let U_i be an induced-universal graph for the family of graphs $\mathcal{F}_i$ for $i = 1, 2, \ldots, k$. Suppose that $\mathcal{H}$ is a family of graphs such that each $H \in \mathcal{H}$ can be decomposed into edge-disjoint spanning subgraphs $H_1, H_2, \ldots, H_k$ where each $H_i \in \mathcal{F}_i$. Then there exists an induced-universal graph W for $\mathcal{H}$ where*

$$|V(W)| \;=\; \prod_{i=1}^{k} |V(U_i)|, \quad \text{and} \quad |E(W)| \;\le\; \sum_{i=1}^{k} |E(U_i)| \prod_{j\neq i} |V(U_j)|^2.$$

Proof. Let the vertices of W be $(u_1, u_2, \ldots, u_k)$ where $u_i \in U_i$ for all i, and there is an edge connecting $(u_1, u_2, \ldots, u_k)$ and $(u'_1, u'_2, \ldots, u'_k)$ if and only if there is an edge connecting u_i to u'_i for some $i = 1, 2, \ldots, k$.

To see that W is our desired graph we start with an $H \in \mathcal{H}$, which by assumption decomposes into $H_1, H_2, \ldots, H_k$. Let $h_i(v)$ denote the vertex that v maps to when H_i is embedded into U_i. Then note that $(h_1(v), h_2(v), \ldots, h_k(v))$ is adjacent to $(h_1(w), h_2(w), \ldots, h_k(w))$ if and only if $h_i(v)$ is adjacent to $h_i(w)$ for some $i = 1, 2, \ldots, k$. This can happen if and only if u and v are adjacent in H_i for some $i = 1, 2, \ldots, k$ which can happen if and only if u and v are adjacent in H. So the map that sends v to $(h_1(v), h_2(v), \ldots, h_k(v))$ is our desired embedding for H.

Finally, the calculation of the number of vertices is immediate, and we note that for fixed i that an edge in $\mathcal{F}_i$ can create at most $\prod_{j\neq i} |V(U_j)|^2$ new edges in $\mathcal{H}$ and the result follows. □

The assumptions that we used in the Theorem are stronger than needed. Namely, the requirement that the H_i be edge-disjoint is not required, it suffices that each edge of H is covered by some H_i. We have stated the proof this way to help make it clear how to generalize the result to multigraphs.

In our case all the $\mathcal{F}_i$ are the same family, namely the graphs on n vertices with maximum degree 2. In such a case we can simplify the above bounds to get the following result.

Corollary 2. *Let U be an induced-universal graph for a family $\mathcal{F}$ of graphs. If every graph in a family $\mathcal{H}$ can be edge-partitioned into k parts, each of which is a graph in $\mathcal{F}$, then there exists an induced-universal graph W for the family $\mathcal{H}$ with*

$$|V(W)| \;\le\; |V(U)|^k, \quad \text{and} \quad |E(W)| \;\le\; k|V(U)|^{2k-2}|E(U)|.$$

Proof. (Proof of Theorem 1) Let G be a graph in the family of graphs on n vertices with maximum degree at most r. By Corollary 1 we know we can decompose G into $\lfloor (r+1)/2 \rfloor$ edge disjoint subgraphs each with maximum degree at most 2 (note some of these graphs might be the empty graph). By Lemma 1 we know that the U given in Figure 1 is an induced-universal graph for the graphs in our decomposition. Applying Corollary 2, with the known size of vertices and edges of U given in Section 3, the result follows. □

For the case when r is even the construction given in Theorem 1 is within a constant factor (depending only on r) of the smallest possible number of vertices. This follows from the observation that the number of induced subgraphs of the induced-universal graph must be at least the number of graphs in the family of $\mathcal{F}$. When nr is even we have for n sufficiently large that,

$$\frac{|V(U)|^n}{n!} \geq \binom{|V(U)|}{n} \geq |\mathcal{F}| \geq e^{-(r^2-1)/4}\Big(\frac{r^{r/2}}{e^{r/2}r!}\Big)^n n^{rn/2}/n!. \tag{1}$$

The end term follows from the known [22] formula for the number of labeled r-regular graphs on n vertices when rn is even, namely there are

$$(1+o(1))\sqrt{2}e^{-(r^2-1)/4}\Big(\frac{r^{r/2}}{e^{r/2}r!}\Big)^n n^{rn/2}$$

such graphs. Simplifying (1) we see that $|V(U)| \geq cn^{r/2}$ for some c depending only on r while the construction used for Theorem 1 has $|V(U)| \leq Cn^{r/2}$ when r is even.

5. Generalizing to multigraphs and directed graphs

The approach given above easily generalizes to other types of graphs. We summarize these results here.

Theorem 4. *Let $\mathcal{F}$ be the family of all multigraphs (i.e., graphs with possible multiple edges or loops) with maximum degree at most r on n vertices. Then there exists an induced-universal multigraph U for $\mathcal{F}$ where*

$$|V(U)| \ \leq\ Cn^{\lfloor (r+1)/2\rfloor}, \quad \text{and} \quad |E(U)| \ \leq\ Dn^{2\lfloor (r+1)/2\rfloor -1},$$

where C and D are constants which depend only on r.

Theorem 4 follows by using the decomposition result of Corollary 1, modifying the graph of Section 3 by adding n isolated loops and $\lfloor n/2\rfloor$ isolated bidirected edges (the graph will have at most $8.5n$ vertices and $9.5n$ edges), and finally modifying the construction of Theorem 3 so that the number of edges between $(u_1,u_2,\ldots,u_k)$ and $(u_1',u_2',\ldots,u_k')$ is $|\{i : u_i \text{ is adjacent to } u_i' \text{ in } U_i\}|$. (While we do not usually work with induced multigraphs we can use them to model induced graphs for graphs with edge weights $1,2,\ldots$ where the degree of a vertex is the sum of the edge weights.)

Theorem 5. *Let $\mathcal{F}$ be the family of all directed (multi)graphs with in-degree and out-degree at most r on n vertices. Then there exists an induced-universal directed (multi)graph U for $\mathcal{F}$ where*

$$|V(U)| \ \leq\ Cn^{r}, \quad \text{and} \quad |E(U)| \ \leq\ Dn^{2r-1},$$

where C and D are constants which depend only on r.

Theorem 5 follows from noting that we can modify the decomposition result given in Section 2 to get the following.

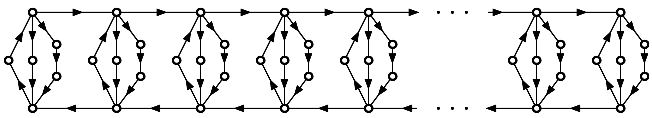

Fig. 3. $\lfloor n/2 \rfloor$ sets of directed bridges connected in serial.

Lemma 2. *Let G be a directed graph with in- and out-degree at most s. Then G can be decomposed into s edge-disjoint directed graphs where the in- and out-degree at each vertex is at most* 1.

We then modify the construction of the graph given in Section 3 by directing the path, adding $\lfloor n/2 \rfloor$ directed two-cycles, directing the three-cycles, removing the four-cycles and replacing the linked bridges with the directed graph given in Figure 3 (the graph will have at most $7n$ vertices and $8.5n$ edges, for the directed mutigraph version we also add n loops). Finally, we modify the construction of Theorem 3 so that there is a directed edge between $(u_1, u_2, \ldots, u_k)$ and $(u'_1, u'_2, \ldots, u'_k)$ if and only if there is a directed edge between u_i and u'_i for some $i = 1, 2, \ldots, k$.

6. Concluding remarks

In this paper we have considered the problem of constructing (sparse) induced-universal graphs for the family of graphs with bounded maximum degree. While for the case when the maximum degree is even we have a construction which is in some sense within a constant of optimal there is still a large gap for the family of graphs with maximum degree odd, namely a factor of $n^{1/2}$. Alon and Capalbo [3] have a construction for an induced universal-graph for the family of graphs with maximum degree at most 3 using $cn^{3/2}$ vertices, though there technique is markedly different from the one presented here. More recently Esperet, Labourel and Ochem [19] were able to use the universal graph of Alon and Capalbo [2] to construct an induced universal graph for odd k with $c_1(k)n^{\lceil k/2\rceil - 1/k}\log^{2+2/k} n$ vertices and $c_2(k)n^{k-2/k}\log^{4+4/k} n$ edges.

While there has been some research in the area of universal graphs large gaps remain. For example, the subject of induced-universal graphs has been little studied, at the same time there has been almost no investigation into (induced-) universal graphs for directed graphs, multigraphs, or hypergraphs. We note that for hypergraphs the techniques of Theorem 3 still hold, the main problem is finding a good decomposition and a corresponding induced-universal graph.

This field is ripe for new ideas and problems.

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