# Method for doing the (approximate) Markov chains for bisection

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Below is an outline of an approach to calculate the (approximate) Markov chains. This is approximate in the sense that it is not a Markov partition so that one part can be either mapped to a single region or equally split between two different regions. Also, we leave out much of the justification, this is only meant to outline the approach.

We start by taking our region representing all triangles and subdivide it into $k^2$ regions by subdividing each edge of this region into $k$ parts. As we go through we will illustrate our techniques with the case $k = 3$. So the subdivision will be as follows.

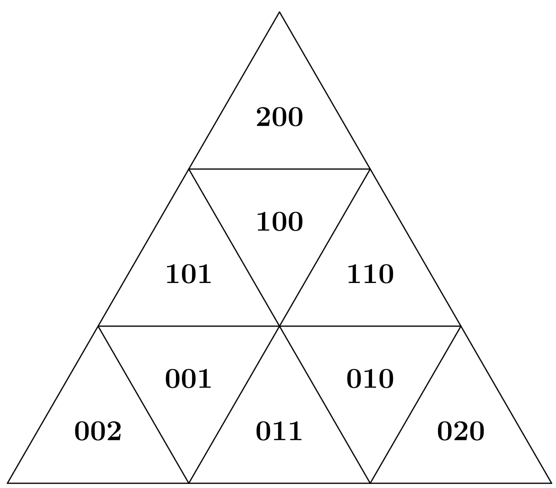


Figure 1: The subdivision in the case $k = 3$ (with labeling).

We have also introduced a labeling in this figure. Each triangle is labeled by three numbers $abc$ where $a$ indicates how far we are from the bottom, $b$ indicates how far we are from the left side and $c$ indicates how far we are from the right side. For convenience we start with the nearest strip to a side as being the "**0**" strip and then increase.

**Observation 1.** *The numbers $a$, $b$ and $c$ are nonnegative and further satisfy $a+b+c = k-1$ or $k-2$. (Dividing the triangles by the sum of their indices will produce a checkerboard-like pattern.)*

For convenience we will want to have a way of converting a label to a number between 0 and $k^2-1$ and vice versa. There are many possible ways to do this, we will do this lexicographically, i.e., in the case of $k=3$ we want **001** $\leftrightarrow 0$, **002** $\leftrightarrow 1$, **010** $\leftrightarrow 2$, **011** $\leftrightarrow 3$, **020** $\leftrightarrow 4$, **100** $\leftrightarrow 5$, **101** $\leftrightarrow 6$, **110** $\leftrightarrow 7$ and **200** $\leftrightarrow 8$.

**Lemma 2.** *The maps between triples $abc$ and numbers $n$ are found by the following two rules ($k$ is the number of subdivisions).*

- *Given $abc$, then $n$ can be found as follows*

$$n = (2ka - a^2) + 2b + \begin{cases} 1 & \text{if } a+b+c = k-1 \text{ and } c > 0; \\ 0 & \text{otherwise.} \end{cases}$$

- *Given $n$, then the triple $abc$ can be found sequentially as follows*

$$\begin{aligned} a &= \lfloor k - \sqrt{k^2 - n} \rfloor \\ b &= \left\lfloor \frac{n + a^2}{2} \right\rfloor - ak \\ c &= \begin{cases} 0 & \text{if } a+b = k-1; \\ n + k - 2 + a^2 - (2k+1)a - 3b & \text{otherwise.} \end{cases} \end{aligned}$$

We now have a convenient labeling, we next need to see how the parts map. To do this we will first scale our region up so that the three vertices are located at $(6k, 0, 0)$, $(0, 6k, 0)$ and $(0, 0, 6k)$ (i.e., as compared to $(\pi, 0, 0)$, $(0, \pi, 0)$ and $(0, 0, \pi)$). The advantage to this is that we can do all the operations as integer operations. The reason that we choose a factor of $6k$ is so that the middle point of each part has *even* coordinates.

**Lemma 3.** *After scaling our big triangle so that the vertices are $(6k, 0, 0)$, $(0, 6k, 0)$ and $(0, 0, 6k)$ then the center of $abc$ is found as follows:*

$$\begin{cases} (6a+2, 6b+2, 6c+2) & \text{if } a+b+c = k-1; \\ (6a+4, 6b+4, 6c+4) & \text{if } a+b+c = k-2. \end{cases}$$

So for instance in our triangle we have that the centers are located as follows **002**: $(2, 2, 14)$; **001**: $(4, 4, 10)$; **011**: $(2, 8, 8)$; **010**: $(4, 10, 4)$; **020**: $(2, 14, 2)$; **101**: $(8, 2, 8)$; **100**: $(10, 4, 4)$; **110**: $(8, 8, 2)$; **200**: $(14, 2, 2)$.

The reason for our setup here is the following observation. (This is the heart of the whole approach!)

**Observation 4.** *If the center of a part maps into the* interior *of another part then the entire part maps into that region. On the other hand if the center of a part maps onto an* edge *then the part gets mapped equally onto the two regions that share that edge.*

For reference, we use the following matrices when computing the bisectors

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

A point (with integer coordinates) will be in the center if and only if *every* coordinate is not divisible by 6. So for instance, if we use the first map on the center point of **010**, which is $(4, 10, 4)$, we get the point $(2, 9, 7)$. To find the part that this lies in we divide each entry by 6 and then throw away any remainder. So in this case we get **011**. It now follows by the observation that under the first mapping that **010** will map into **011**.

A point (with integer coordinates) will be on an edge if and only if there is *one* (and it turns out to be exactly one) coordinate which is divisible by 6. So for instance if we use the first map on the center point of **011**, which is $(2, 8, 8)$, we get the point $(1, 12, 5)$. To find the two parts that this straddles we again divide each entry by 6 and throw away any remainder, but now the entry which is exactly divisible by 6 can also go down one in value. So in this case, division by 6 gives **020** but we also can decrease the $b$ coordinate by 1 and get **010**. It now follows by the observation that under the first mapping that **011** will map half onto **020** and half onto **010**.

The rest is now straightforward. Starting from 0 to $k^2 - 1$ we find the region that the number corresponds to, see where it maps under each of the six possible maps (either into a single region or straddling two regions). Once we know where it maps we find the number(s) corresponding to the region(s) it mapped to and then update our transition probability matrix accordingly. In our case we get the following matrix

$$\begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

We now want to find the *left* eigenvector of this matrix corresponding to the eigenvalue of 1 (in Maple we would find the eigenvector of the transpose). For the above matrix, this eigenvector is $[1, 1/2, 1, 1, 1/2, 1, 1, 1, 1/2]$ (scaled so that the largest entry is 1). Putting this in the triangle from the beginning we get the following.

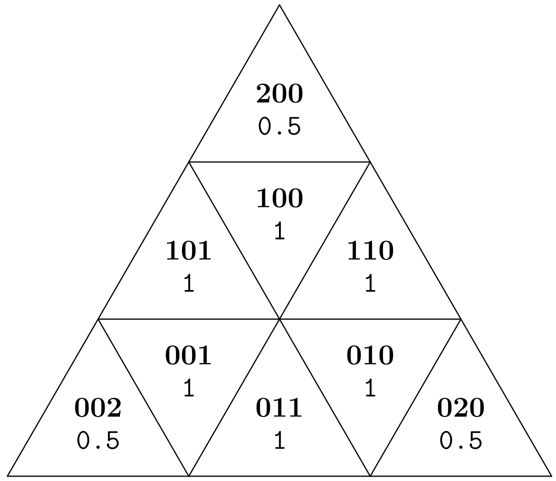


Figure 2: The value of the eigenvector in the different parts.

Using these values in the different regions we can then shade with darker regions being more likely. So for the $k = 3$ case we end up with the following picture.

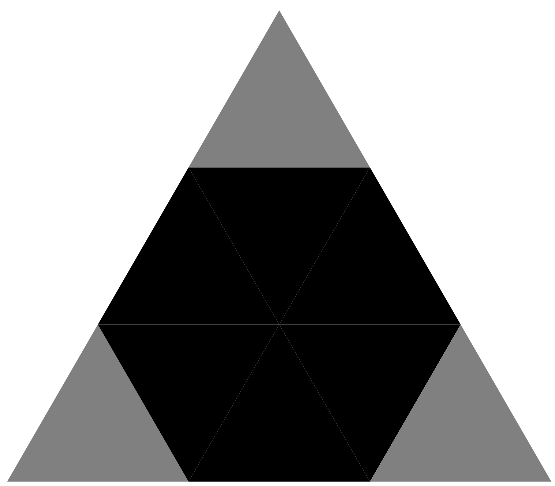

Figure 3: Using the eigenvector to shade the different regions.

Repeating the same thing with $k = 7$ we get the following pictures.

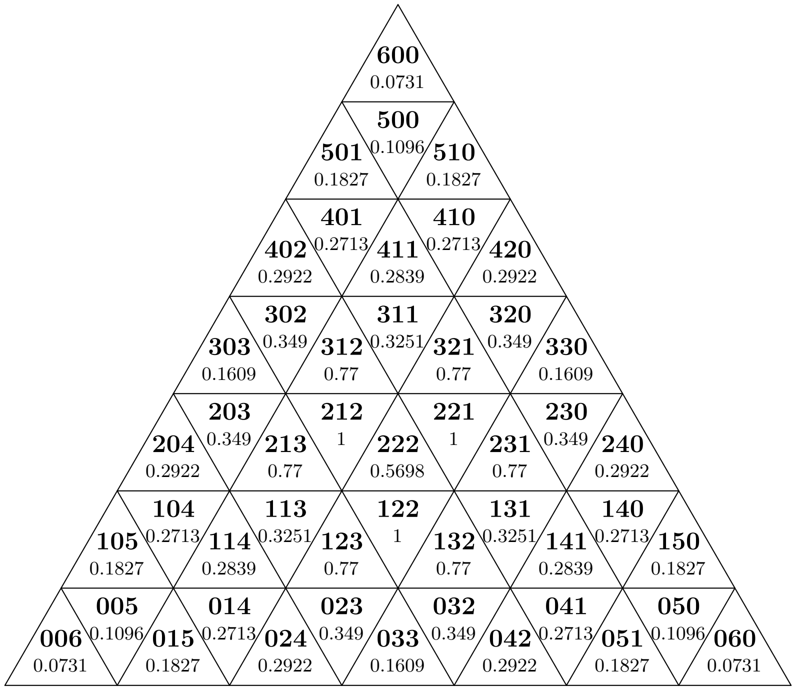


Figure 4: The value of the eigenvector in the different parts for $k = 7$.

Figure 5: The shaded triangle for $k = 7$.