Tiling polygons with lattice triangles

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(joint work with Fan Chung, Ron Graham and Miklós Laczkovich)

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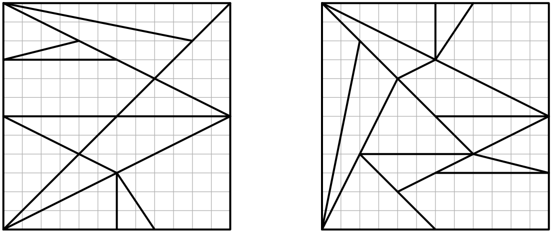
Stomachion of Archimedes

The Stomachion of Archimedes is a collection of 14 polygons that can tile a 12×12 square. Fan Chung and Ron Graham and independently Bill Cutler showed that there are essentially 268 ways to use these pieces to tile a square.



Fact: Every tiling of the 12×12 square with the Stomachion pieces has all vertices with integer coordinates.

Question

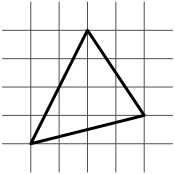
If we tile a large polygon with Stomachion pieces (assuming unlimited supplies of each piece), what can we say about the location of the vertices?

Question

If we tile a large polygon with some collection of lattice triangles (assuming unlimited supplies of each piece), what can we say about the location of the vertices?

Lattice triangles

A lattice triangle is a triangle which can be placed in the plane with vertices at integer coordinates.



All right triangles with integer leg lengths are lattice triangles.

Fact: A triangle has all cotangents rational *if and only if* it is similar to a lattice triangle.

Vertices are rational

Theorem (Laczkovich)

If a simple polygon P is tiled by triangles $\Delta_1, \ldots, \Delta_n$, then the location of the vertices are in the field generated by the coordinates of the vertices of P and the cotangents of the triangles Δ_j.

Corollary

If a simple polygon P with rational coordinates is tiled using lattice triangles then all of the vertices are rational.

Main result

Theorem

Let P be a simply polygon whose vertices are rational coordinates, one vertex is the origin with an adjacent vertex on the x-axis. Suppose that P is dissected into congruent copies of lattice triangles $\Delta_1, \Delta_2, \ldots, \Delta_k$ and that $\alpha_1, \alpha_2, \ldots, \alpha_{3k}$ are the angles of the triangles with $\cot \alpha_i = a_i/b_i$. Then the coordinates of the internal vertices are of the form

$$\frac{p}{\prod_{i=1}^{3k}(a_i^2 + b_i^2)^{n_i}}, \qquad \text{with } p, n_i \in \mathbb{Z}.$$

Useful fact

If $\cot\alpha_j = \dfrac{a_j}{b_j}$ then

$$|\sin\alpha_j| = \frac{1}{\sqrt{1+\cot^2\alpha_j}} = \frac{b_j}{\sqrt{a_j^2+b_j^2}} \text{ and}$$

$$|\cos\alpha_j| = \frac{1}{\sqrt{1+\tan^2\alpha_j}} = \frac{a_j}{\sqrt{a_j^2+b_j^2}}.$$

Fact: If $\theta = \pm t_1\alpha_1 \pm t_2\alpha_2 \pm \cdots \pm t_{3k}\alpha_{3k}$ then

$$\cos\theta = \frac{u\sqrt{r}}{\prod_{i=1}^{3k}(a_i^2+b_i^2)^{t_i}} \quad \text{and} \quad \sin\theta = \frac{v\sqrt{s}}{\prod_{i=1}^{3k}(a_i^2+b_i^2)^{t_i}}.$$

Sketch of proof of main result

- We already know that all vertices are rational, only need to show of the correct form.
- We can get from any vertex of P to $(0,0)$ by moving along edges of the triangles in the tiling of P.
- For an internal vertex fix a path to the origin. We now proceed by induction on the path to show each vertex has the correct form.
  - Trivially holds for $(0,0)$.
  - Between (x_i, y_i) and (x_{i+1}, y_{i+1}) we move along an edge of triangle (length $\sqrt{a}$, where a an integer) and the angle is of the form $\theta = s_1\alpha_1 + s_2\alpha_2 + \cdots + s_{3k}\alpha_{3k}$. Therefore

$$x_{i+1} = x_i + \sqrt{a}\cos\theta = x_i + \frac{u\sqrt{ar}}{\prod_{i=1}^{3k}(a_i^2 + b_i^2)^{s_i}}.$$

  - Since x_i and x_{i+1} are rational can conclude that $\sqrt{ar}$ is an integer and so x_{i+1} is of the desired form. (Similar argument for y_{i+1}.) $\square$

Possible primes in the denominators

Since the terms in the denominators are of the form $a_i^2 + b_i^2$ then the only possible primes are those which can divide $a_i^2 + b_i^2$, i.e., 2 and primes of the form $4k + 1$.

Theorem

Let P be a simple polygon in the plane with the vertices of the coordinates all rational and P, one vertex is at the origin and P is dissected into lattice triangles. Then the denominators of the coordinates of the vertices are odd.

(The proof is similar to the previous result.)

We have seen that if the coordinates are rationals that the denominators must be of a particular form... but is it even possible that we can build up large denominators?

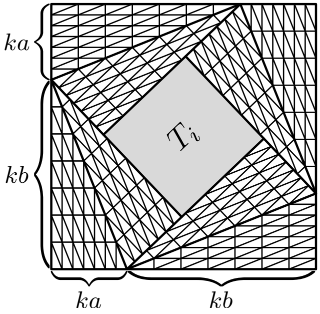
Looking at the proof of our main result the large denominators come when we have $\theta = \pm t_1\alpha_1 \pm t_2\alpha_2 \pm \cdots \pm t_{3k}\alpha_{3k}$ with the t_i large.

We will use this idea to show how to build up large denominators in the special case when all of the lattice triangles are right triangles.

An iterative tiling of large(!) squares with right triangles

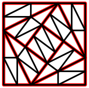
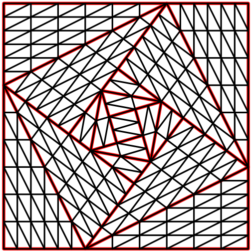
To create an angle of $\theta = 2s_1\beta_1 + 2s_2\beta_2 + \cdots + 2s_k\beta_k$ (here β_i is the smallest angle in the right triangle) iteratively build up squares by layers.

- T_0 will be a tiling of *any* square using right triangles.
- Given T_i we put it inside of a larger square as follows:

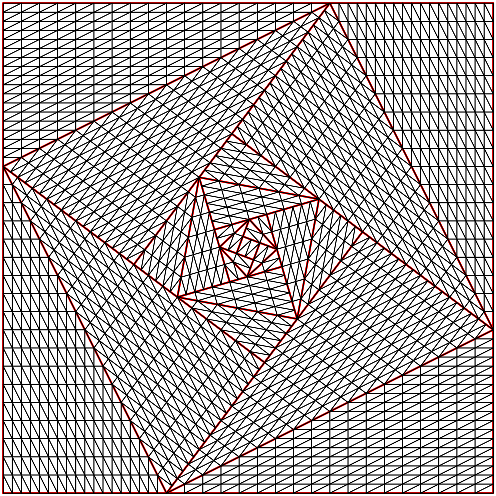
$T_{i+1} =$ 

An example

Tiling using triangles with side lengths 1, 2 and $\sqrt{5}$.

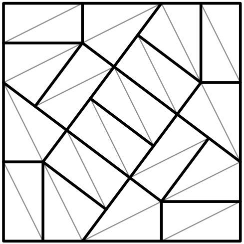
$T_0 =$  $T_1 =$  $T_2 =$ 

T_0 a tiling of $[0,2]\times[0,2]$ has all integer coordinates.
T_1 a tiling of $[0,6]\times[0,6]$ has a coordinate at $(16/5, 8/5)$.
T_2 a tiling of $[0,18]\times[0,18]$ has a coordinate at $(256/25, 208/25)$.

$T_3 =$ 

T_3 a tiling of $[0,54]\times[0,54]$ has a coordinate at $(3536/125, 3448/125)$.

Boxes and kites

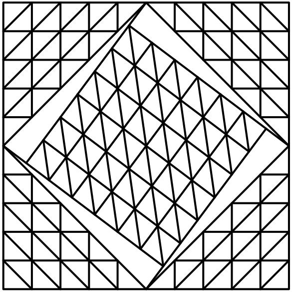


If we look at the tilings of the T_i we see that they consist of essentially two objects: boxes and kites.

When we are packing with right triangles and the hypotenuses are linearly independent (over $\mathbb{Q}$) then we will always be in the case of packing with boxes and kites.

Example without kites and boxes

When we do not have linear independence more interesting situations can occur.

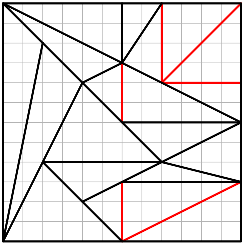


(A packing of $[0,10]\times[0,10]$ with 172 of the 1-1-$\sqrt{2}$ triangles and 4 of the 1-7-$\sqrt{50}$ triangles.)

Returning to the Stomachion

The Stomachion has lattice polygons. So to apply our results we first subdivide all the polygons into lattice triangles.

Any tiling using the Stomachion pieces is a subset of tiling using these lattice triangles.



Fact: Any tiling of a polygon with rational coordinates (one vertex at $(0,0)$ with an adjacent vertex on the x axis) using pieces of the Stomachion has coordinates of vertices of the form

$$\frac{p}{5^q 13^r 17^s}.$$