Problems proposed by Steve Butler

Mathematics Magazine **76** (April 2003), 151.
1668. Let f be a real valued function defined on an open interval containing $[a,b]$. Assume that f has a continuous second derivative on I and there is a single line tangent to the graph of $y=f(x)$ at $(a,f(a))$ and $(b,f(b))$. Prove that if $f''(x)$ is not identically zero on (a,b), then $f''(x)$ must change sign at least twice on (a,b).

Solution published in *Mathematics Magazine* **77** (April 2004), 159.

Alternative solution:
There is a single line tangent at $(a,f(a))$ and $(b,f(b))$ if and only if $f'(a)=f'(b)$ (same slope) and $f(a)-af'(a)=f(b)-bf'(b)$ (same y-intercept). Since

$$\begin{aligned}\int_a^b (Cx+D)f''(x)\,dx &= \big(C(f(x)-xf'(x))+Df'(x)\big)\Big|_a^b\\ &= C\big((f(b)-bf'(b))-(f(a)-af'(a))\big)+D\big(f'(b)-f'(a)\big),\end{aligned}$$

it follows that there is a single tangent line at $(a,f(a))$ and $(b,f(b))$ if and only if $\int_a^b(Cx+D)f''(x)\,dx=0$ for all constants C and D.

If f'' never changed sign on (a,b) then since $f''\not\equiv 0$ we have $\int_a^b f''(x)\neq 0$ and so we could not have the tangent line. If f'' changed sign only once then for some $a<c<b$ we would have $(x-c)f''(x)$ either nonnegative or nonpositive on (a,b), in either case we would again have that $\int_a^b(x-c)f''(x)\,dx\neq 0$ and so again we could not have the tangent line. So it must be that f'' changes sign at least twice.

The American Mathematical Monthly **110** (Aug.-Sep. 2003), 637.
11030. Show that for $d<-1$ there are exactly two real-valued functions f such that for all real x and y $f(x+y)-f(x)f(y)=d\sin x\sin y$.

Solution published in *The American Mathematical Monthly* **112** (April 2005), 371.

More information at `http://www.math.ucsd.edu/~sbutler/PDF/cauchy.pdf`

Mathematics Magazine **78** (October 2005), 325.
1730. Let A and B be symmetric, positive semi-definite matrices such that $A+B$ is positive definite, and let $\|\mathbf{y}\|$ denote the usual 2-norm of the vector $\mathbf{y}$. Prove that for all $\mathbf{x}\neq\mathbf{0}$,

$$\big\|(I-A)(I+A)^{-1}(I-B)(I+B)^{-1}\mathbf{x}\big\|<\|\mathbf{x}\|.$$

Solution published in *Mathematics Magazine* **79** (October 2006), 313-314.

More information at `http://www.math.ucsd.edu/~sbutler/PDF/pos_norm.pdf`

The American Mathematical Monthly **114** (January 2007), 77.
11265. (*Proposed jointly with Jia Mao.*) For a graph G, let $c(G)$ denote the minimum k such that every edge lies in a cycle of length at most k ($c(G)$ is infinite when G has a cut-edge). Prove that for $n \geq 3$ the minimum of $|E(G)| + c(G)$, taken over all n-vertex connected graphs, is $n + \lceil 2\sqrt{n-1} \rceil$.

Solution published in *The American Mathematical Monthly* **116** (February 2009), 180-181.

Mathematics Magazine **80** (February 2007), 77.
1761. For integer $n \geq 2$ define the sets

$$A(n) = \{(k,l) : 1 \leq k \leq l \leq n,\ k+l \leq n, \text{ and } \gcd(k,l) = 1\}$$

$$B(n) = \{(k,l) : 1 \leq k \leq l \leq n,\ k+l > n, \text{ and } \gcd(k,l) = 1\},$$

where $\gcd(k,l)$ denotes the greatest common divisor of the integers k and l. Prove that $A(n)$ and $B(n)$ have the same cardinality.

Two different solutions published in *Mathematics Magazine* **81** (February 2008), 64-66.