An interstice relationship for flowers with four petals

Steve Butler, Ron Graham, Gerhard Guettler and Colin Mallows

Abstract. Given three mutually tangent circles with bends (or the reciprocal of the radius) a, b and c respectively, an important quantity associated with the triple is the value $\langle a,b,c\rangle := ab+ac+bc$. In this note we show in the case when a central circle with bend b_0 is "surrounded" by four circles, i.e., a flower with four petals, with bends b_1, b_2, b_3, b_4 that either

$$\sqrt{\langle b_0,b_1,b_2\rangle}+\sqrt{\langle b_0,b_3,b_4\rangle}=\sqrt{\langle b_0,b_2,b_3\rangle}+\sqrt{\langle b_0,b_4,b_1\rangle}$$

or

$$\sqrt{\langle b_0,b_1,b_2\rangle}=\sqrt{\langle b_0,b_2,b_3\rangle}+\sqrt{\langle b_0,b_3,b_4\rangle}+\sqrt{\langle b_0,b_4,b_1\rangle}$$

(where $\langle b_0,b_1,b_2\rangle$ is chosen to be maximal).

As an application we give a sufficient condition for the alternating sum of the $\sqrt{\langle a,b,c\rangle}$ of a packing in standard position to be 0. (A packing is in standard position when we have two circles with bend 0, i.e., parallel lines, and the remaining circles are packed in between.)

Keywords. Interstice, Apollonian, petals.

1. Introduction

The packing of circles has connections to analysis, geometry, combinatorics, number theory, algebra and art [8, 9, 10]. One of the most studied forms of circle packings are Apollonian circle packings which can be formed by starting with three mutually tangent circles, which form a curvilinear triangle (also called interstice). Then one places a new circle into the curvilinear triangle thereby producing three new curvilinear triangles, and we can repeat this process indefinitely [4, 5, 6, 7].

A remarkable property of Apollonian packings is that if the bends (which are the reciprocal of the radii) of the initial three circles are integers and if

$$\langle a,b,c\rangle := ab+ac+bc$$

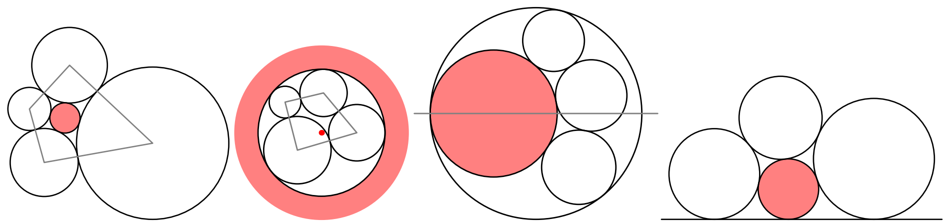


Figure 1: Configurations where (1) holds (the circle with bend b_0 is shaded red).
(a) All bends are positive. The center of the shaded circle lies inside the quadrilateral formed by the centers of the other four circles.
(b) The central circle has negative bend. The center of the shaded circle lies inside the quadrilateral formed by the centers of the other four circles.
(c) A noncentral circle has negative bend. Two of the centers lie on one side of the line through the center of the central circle and the circle with negative bend.
(d) A noncentral circle has bend 0.

is a square, then each subsequent circle will also have integer bend and further for any three mutually tangent circles with bends d, e, f then $\langle d, e, f\rangle$ will be square. Note that a straight line is a circle with bend zero.

Most of the research about Apollonian circle packings (and circle packings in general) have focused mostly on the bends. In this note we will focus on $\langle a, b, c\rangle$, or more precisely $\sqrt{\langle a, b, c\rangle}$. Our main result deals with the situation when a central circle is "surrounded" by four circles all tangent to the central circle and to their two neighbors, sometimes called a flower with four petals.

Theorem 1. *Given a circle with bend b_0 suppose that there are four circles with bends b_1, b_2, b_3, b_4 which are all tangent to the circle with bend b_0; b_1 is tangent to b_2 and b_4; b_2 is tangent to b_1 and b_3; b_3 is tangent to b_2 and b_4; and b_4 is tangent b_3 and b_1. Further, all circles have disjoint interiors (the "interior" of a circle with negative bend includes the point at infinity). Then either*

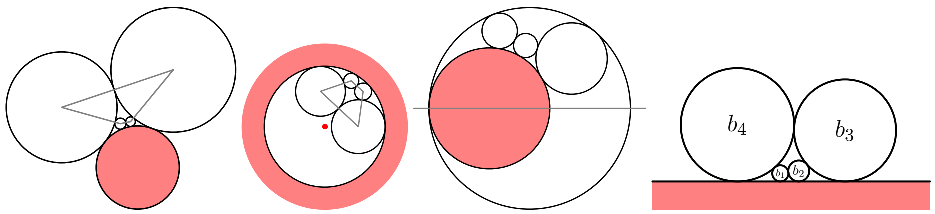
$$\sqrt{\langle b_0, b_1, b_2\rangle} + \sqrt{\langle b_0, b_3, b_4\rangle} = \sqrt{\langle b_0, b_2, b_3\rangle} + \sqrt{\langle b_0, b_4, b_1\rangle}, \qquad (1)$$

or

$$\sqrt{\langle b_0, b_1, b_2\rangle} = \sqrt{\langle b_0, b_2, b_3\rangle} + \sqrt{\langle b_0, b_3, b_4\rangle} + \sqrt{\langle b_0, b_4, b_1\rangle}, \qquad (2)$$

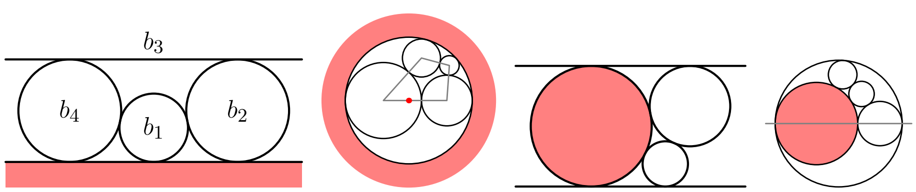
(where b_1, b_2 are chosen to maximize $\langle b_0, b_1, b_2\rangle$).

We can also characterize the different situations for which these equations hold. For instance Figure 1 shows configurations where (1) holds, Figure 2 shows configurations where (2) holds and Figure 3 shows configurations where both equations hold.



(a) $b_i > 0$. (b) $b_0 < 0$. (c) $b_4 < 0$. (d) $b_0 = 0$.

Figure 2: Configurations where (2) holds (the circle with bend b_0 is shaded red).
(a) All bends are positive. The center of the shaded circle lies outside the quadrilateral formed by the centers of the other four circles.
(b) The central circle has negative bend. The center of the shaded circle lies outside the quadrilateral formed by the centers of the other four circles.
(c) A noncentral circle has negative bend. Three of the centers lie on one side of the line through the centers of the central circle and the circle with negative bend.
(d) The central circle has bend 0.



(a) $b_0 = b_1 = 0$. (b) $b_0 < 0$. (c) $b_0 > 0, b_3 = b_4 = 0$. (d) $b_4 < 0$.

Figure 3: Configurations where both (1) and (2) hold, i.e., $\langle b_0, b_3, b_4 \rangle = 0$.
(a) $b_0 = b_3 = 0$. In this case $\langle b_0, b_2, b_3 \rangle = \langle b_0, b_3, b_4 \rangle = 0$.
(b) $b_0 < 0$. The center of the central circle lies on the line through the centers of the circles with bends b_3 and b_4.
(c) $b_3 = b_4 = 0$. In this case $b_1 b_2 = 4b_0^2$.
(d) $b_4 < 0$. The centers of the circles with bends b_0, b_3 and b_4 are collinear.

We will give two proofs for Theorem 1. In Section 2 we will introduce some basic properties of inversions and bends and give a geometrical proof of the result. In Section 3 we will give a different proof based on algebraic tools. Finally, in Section 4 we will introduce G-packings and use the Theorem 1 to give a sufficient condition for a configuration to be a G-packing.

2. Geometrical proof of Theorem 1

Some basic facts about inversions

Before we begin our proof we first need to introduce some basic facts about inversion. Fix a circle with radius k and map points at distance s to distance t from the center (preserving the same direction) such that $st = k^2$ (the center of the circle goes to ∞ and ∞ maps to the center of the circle). We note that this map is conformal (preserves angles). The circle of inversion is fixed under this map and applying the map twice is the identity. In addition, circles map to circles (where a straight line is a circle with center at ∞).

The following observations will be helpful in the ensuing discussion.

Observation 1. *Given a point with a line through that point and another point there is a unique circle that contains the two points and for which the line is a tangent line to the circle.*

This simply follows by noting that the center of the circle must lie on the perpendicular of the line through the point as well as the perpendicular bisector of the line segment joining the two points. Once we have the center the circle is easily found. In the case that the two points both lie on the given line then the desired circle is simply the line.

Observation 2. *Circle A is fixed under inversion in circle T if and only if either $A = T$ or A and T intersect at a right angle.*

This follows by noting that if A and T do not intersect then either A lies outside of T and under inversion goes to the inside or vice-versa. So if A is fixed then A and T must intersect. If they only intersect at one point then a similar argument shows again A cannot be fixed. If they intersect at three or more points then $A = T$. Finally, if they intersect at two points then since it is a conformal map the angle is preserved under the inversion which is only possible if the angle of intersection is a right angle. Conversely if $A = T$ then clearly A is fixed, while if A and T intersect at a right angle then there is a second point of intersection and these two points of the circle as well as the tangent line are fixed under the inversion and so by the previous observation all of A is fixed.

As an application we have the following lemma which will form the heart of our geometric approach.

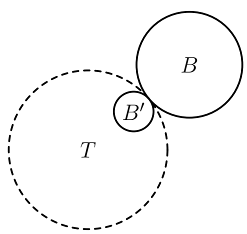
Lemma 1. *Given three circles A, B and C such that B is tangent to A and to C (with different points of tangency), then there is a unique circle T so that inverting in the circle T fixes the circles A, B and C.*

Proof. Construct the tangent lines to B at the points of tangency. Let T be the unique circle whose center is the intersection of these lines and contains the points of tangency. Then by the previous observation the circle T intersects A, B and C at right angles and so they are all fixed under the inversion.

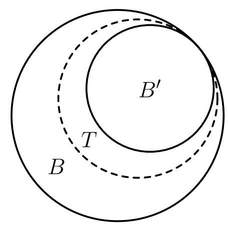
In the case that the tangent lines are parallel then the desired T is the straight line joining the two points of tangency. Finally if the perpendicular lines are the same (which will happen if B is a straight line), the desired line is the circle centered halfway between the points of tangency that contains the two points of tangency. □

We also need to know how the radii and bends of the circle are effected under inversion. This is done with the help of the following two lemmas.

Lemma 2. *Let T be a circle with radius $1/a$, let B be a circle with radius $1/b$ tangent to T (lying in the "exterior" of T), and let B' be the circle of B under inversion through T (so it lies in the "interior" of T). If T lies in the exterior of B (Figure 4a), then B' has radius $1/(2a+b)$. While if T lies in the interior of B (Figure 4b), then B' has radius $1/(2a-b)$.*



(a) T in exterior of B.



(b) T in interior of B.

Figure 4: The two possibilities for Lemma 2 (circle T is dashed).

Proof. Since the point of tangency is fixed under inversion it suffices to consider where the furthest point of B maps to. In the case that T is in the exterior of B we have that the furthest point is at distance $2/b + 1/a = (2a + b)/ab$ from the center of T. Under inversion this will map to a point at distance $b/\big(a(2a + b)\big)$ from the center of T. In particular, the radius of the circle B' will be

$$\frac{1}{2}\left(\frac{1}{a} - \frac{b}{a(b+2a)}\right) = \frac{1}{2a+b}.$$

In the case that T is in the interior of B we have that he furthest point is at distance $2/b - 1/a = (2a - b)/ab$ from the center of T. Under inversion this will map to a point at distance $b/\big(a(2a - b)\big)$ from the center of T. In particular, the radius of the circle B' will be

$$\frac{1}{2}\left(\frac{1}{a} + \frac{b}{a(2a-b)}\right) = \frac{1}{2a-b}.$$ □

The preceding lemma is expressed in terms of radii, but has been stated in a way to show conections to the bends. Namely, if we are in the case in Figure 4a then we expect $a, b > 0$ (to have disjoint interior) and so the bend of B' would be $b' = 2a + b$. On the other hand if we are in the case in Figure 4a then we expect

$a > 0$ and $b < 0$ (again to have disjoint interior) and so we would work with $-b$ instead of b so that the bend of B' would again be $b' = 2a + b$. So expressed as *bends* there is no distinction between the two cases, however we will see that the difference in the two cases when expressed as *radii* will lead to the two different cases in Theorem 1.

Lemma 3. *Let A, B, C be three mutually tangent circles with bends a, b, c (with disjoint interiors), and let a', b', c' be the bends of the circles after inverting in A. Then $\langle a, b, c\rangle = \langle a', b', c'\rangle$.*

Proof. First we note that when we invert in A we have that $a' = -a$. Without loss of generality we may assume that $a > 0$. The basic situation is shown in Figure 5 which shows the case when $b, c > 0$. By Lemma 2 we have that $b' = 2a + b$ and $c' = 2a + c$ and so we have

$$\begin{aligned}\langle a', b', c'\rangle &= \langle -a, 2a + b, 2a + c\rangle = -a(2a + b) - a(2a + c) + (2a + b)(2a + c)\\ &= -2a^2 - ab - 2a^2 - ac + 4a^2 + 2ac + 2ab + bc = ab + ac + bc = \langle a, b, c\rangle.\end{aligned}$$

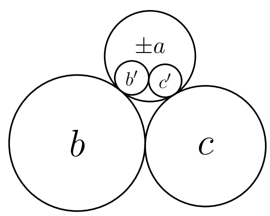


Figure 5: A basic case for Lemma 3.

Another situation (not shown) is when one of B or C has negative bend (they cannot both have negative bend since the interiors are disjoint), but following the discussion preceding the lemma we will again have that $a' = -a$, $b' = 2a + b$ and $c' = 2a + c$ and so the result still holds. □

Orthogonal circles and $\sqrt{\langle a, b, c\rangle}$

Our last step before we give the first proof of Theorem 1 is to give an interpretation for $\sqrt{\langle a, b, c\rangle}$.

Lemma 4. *Let A, B and C be three mutually tangent circles with disjoint interiors and bends a, b and c respectively. Then the (unique) circle going through the three points of tangency has (positive) bend $\sqrt{\langle a, b, c\rangle}$.*

Proof. We first observe that the circle which goes through the three points of tangency intersects the circles A, B and C at right angles (hence we can call these orthogonal circles). One way to see this is to note that we can translate any three mutually tangent circles to any other three mutually tangent circles, via a properly chosen inversion. So for instance we may assume that A' is $y = 0$, B' is $(x-1)^2 + (y-1)^2 = 1$ and C' is $(x+1)^2 + (y-1)^2 = 1$. So the three points of tangency are $(0,1)$, $(1,0)$ and $(-1,0)$ showing the desired circle is $x^2 + y^2 = 1$. It is easy to see that this intersects at right angles the three circles. Finally we can invert back and recall that angles are preserved under inversion.

Let us consider the case when $a, b, c > 0$, then if we connect the centers of the three circles then the circle going through the points of tangency is the incircle of the resulting triangle. The semi-perimeter of this triangle is $s = 1/a + 1/b + 1/c$ and so the radius of the incircle is

$$r = \sqrt{\frac{\left(s - \frac{1}{a} - \frac{1}{b}\right)\left(s - \frac{1}{b} - \frac{1}{c}\right)\left(s - \frac{1}{c} - \frac{1}{a}\right)}{s}} = \sqrt{\frac{1}{ab + ac + bc}} = \frac{1}{\sqrt{ab + ac + bc}}.$$

Showing that the bend of the incircle is $\sqrt{\langle a, b, c\rangle}$.

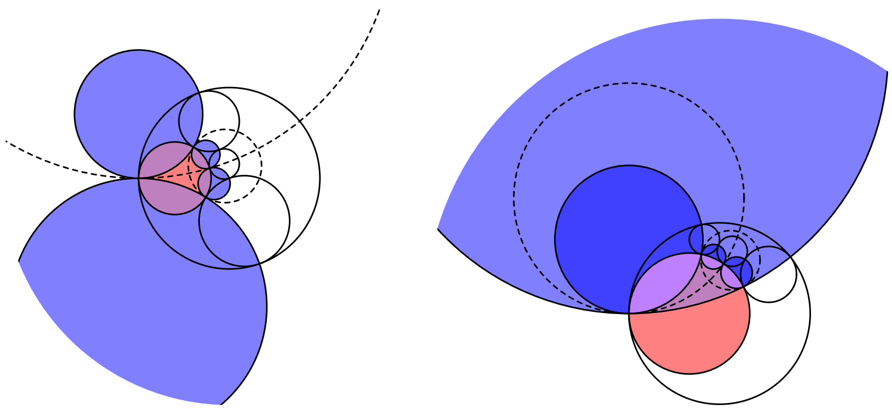
In the case that one of the circles has negative bend (suppose it is a) we note that the circle which passes through the three points of tangency will be unchanged by inverting about A, further the circle will still pass through the three points of tangency of the inverted circles. By the previous case we can conclude this circle has bend $\sqrt{\langle a', b', c'\rangle} = \sqrt{\langle a, b, c\rangle}$ (using Lemma 3).

If two of the circles has 0 bend then the circle passing through the points of tangency is a straight line and has bend $0 = \sqrt{\langle 0, 0, c\rangle}$. Finally, if one of the circles has bend 0 (suppose it is a) then the desired circle is centered halfway between the points of tangency of B and C. An easy exercise shows that these points of tangency are at distance $2/\sqrt{bc}$ so that the circle has bend $\sqrt{bc} = \sqrt{\langle 0, b, c\rangle}$. □

We are now ready to give the first proof of Theorem 1. Starting with our circles b_i we construct the four orthogonal circles passing through the points of tangency between the circles with bends b_0, b_i, b_{i+1}. By the preceding lemma these will have (positive) bends $\sqrt{\langle b_0, b_1, b_2\rangle}$, $\sqrt{\langle b_0, b_2, b_3\rangle}$, $\sqrt{\langle b_0, b_3, b_4\rangle}$, $\sqrt{\langle b_0, b_4, b_1\rangle}$. Now construct the circle which fixes the circles with bends b_0, b_1, b_3 under inversion, and similarly the circle which fixes the circles with bends b_0, b_2, b_4 (these exist by Lemma 1).

Observe that the orthogonal circles will be fixed under inversion about either of the inverting circles we have drawn. Further, the two inverting circles divides the plane up into four regions and there will be exactly one orthogonal circle in each region. There are now two situations which can occur, either the orthogonal circle in the unbounded region does not contain the other orthogonal circles (as shown in Figure 6a, the situations where this can occur are shown in Figure 1), or the orthogonal circle in the unbounded region does contain the other orthogonal circles (as shown in Figure 6b, the situations where this can occur are shown in Figure 2).

In the case when the orthogonal circle in the unbounded region does not contain the other orthogonal circles we can pair circles up across one of the inverting circles. If the inverting circle has bend $\beta > 0$ then (for some appropriate labeling



(a) Based on Figure 1c.

(b) Based on Figure 2c.

Figure 6: The circle with bend b_0 is shaded red, the orthogonal circles are shaded blue, and the inversion circles which fix the diagram are dashed.

of the bends) we have the following two relationships by Lemma 2 (substituting in the expressions for the bends of the orthogonal circles):

$$\sqrt{\langle b_0, b_1, b_2\rangle} = 2\beta + \sqrt{\langle b_0, b_2, b_3\rangle},$$
$$\sqrt{\langle b_0, b_3, b_4\rangle} = 2\beta + \sqrt{\langle b_0, b_4, b_1\rangle}.$$

Now taking the difference of these two gives (1).

In the case when the orthogonal circle in the unbounded region does contain the other orthogonal circles we can still pair circles up across one of the inverting circles. If the inverting circle has bend $\beta > 0$ then (for some appropriate labeling of the bends) we have the following two relationships by Lemma 2 (substituting in the expressions for the bends of the orthogonal circles):

$$\sqrt{\langle b_0, b_1, b_2\rangle} = 2\beta + \sqrt{\langle b_0, b_2, b_3\rangle},$$
$$\sqrt{\langle b_0, b_3, b_4\rangle} = 2\beta - \sqrt{\langle b_0, b_4, b_1\rangle}.$$

Now taking the difference of these two gives (2).

Finally, we note that for the configurations shown in Figure 3 is when the circle in the unbounded region has bend 0, i.e., the orthogonal circle is a line.

3. Algebraic proof of Theorem 1

We now give a second proof of Theorem 1. Here the main idea is to express $\langle a, b, c\rangle$ as the square of something, so that taking square roots becomes simple.

We first handle the cases where the central circle has bend 0. We are either in the case shown in Figure 3a or Figure 2d. For the configuration shown in Figure 3a this reduces to $\sqrt{b_1 b_2} = \sqrt{b_4 b_1}$ which is trivially true by symmetry. For the configuration shown in Figure 2d this follows from the already used fact that if b_i and b_j are tangent to each other and the circle with bend 0 then the distance between the points of tangency on the circle with bend 0 is $2/\sqrt{b_i b_j}$, which implies

$$\frac{2}{\sqrt{b_4 b_3}} = \frac{2}{\sqrt{b_4 b_1}} + \frac{2}{\sqrt{b_1 b_2}} + \frac{2}{\sqrt{b_2 b_3}}.$$

Now mutliply both sides by $\sqrt{b_1 b_2 b_3 b_4}/2$ to get the result.

When the bend of the central circle is not 0 we can translate and scale so that it is a circle with radius 1 centered at the origin. The remaining circles are now either outside the circle of radius 1 (so $b_0 = 1$) or inside the circle of radius 1 (so $b_0 = -1$). We will work through the case $b_0 = -1$. The other case follows by inverting around the central circle and noting that by Lemma 3 any relationship between the interstices will still be preserved.

Instead of using b_1, b_2, b_3, b_4 we now will use a, b, c, d. We may assume $a \geq c > 1$ and a has its center on the x-axis and is tangent to the circle with bend 1 at $(1, 0)$, so that we have the arrangement as shown in Figure 7 where θ is the angle between the centers of the circles with bends a and c.

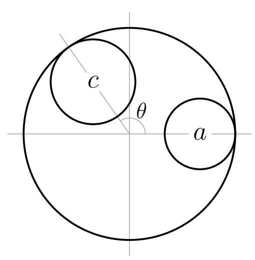


Figure 7: Arrangement of circles for the proof of Theorem 1.

We now express the bends b and d in terms of a, c and θ. Since the circles with bends b and d are both tangent to the same set of three circles then their bends both satisfy the following three equations (where γ is the bend and (x, y)

the center of the circle).

$$\begin{aligned} x^2+y^2 &= \Big(1-\frac{1}{\gamma}\Big)^2 \\ \Big(x-(1-\frac{1}{a})\Big)^2+y^2 &= \Big(\frac{1}{a}+\frac{1}{\gamma}\Big)^2 \\ \Big(x-(1-\frac{1}{c})\cos\theta\Big)^2+\Big(y-(1-\frac{1}{c})\sin\theta\Big)^2 &= \Big(\frac{1}{c}+\frac{1}{\gamma}\Big)^2 \end{aligned}$$

There are two solutions for γ which give us the two bends b and d, it doesn't matter for our purposes which is which and so we will let

$$\begin{aligned} b &= 1+\frac{2\big(a+c-2+\sqrt{Q}\big)}{(a-1)(c-1)(1-\cos\theta)}, \\ d &= 1+\frac{2\big(a+c-2-\sqrt{Q}\big)}{(a-1)(c-1)(1-\cos\theta)}. \end{aligned}$$

where $Q = 2(a-1)(c-1)(1+\cos\theta)$. Now (with a lot of simplifying) we have the following

$$\begin{aligned} 2(a-1)(c-1)(1-\cos\theta)\langle -1,a,b\rangle &= 4(a-1)^2+4(a-1)\sqrt{Q}+Q \\ &= \big(2(a-1)+\sqrt{Q}\big)^2 \\ 2(a-1)(c-1)(1-\cos\theta)\langle -1,b,c\rangle &= 4(c-1)^2+4(c-1)\sqrt{Q}+Q \\ &= \big(2(c-1)+\sqrt{Q}\big)^2 \\ 2(a-1)(c-1)(1-\cos\theta)\langle -1,c,d\rangle &= 4(c-1)^2-4(c-1)\sqrt{Q}+Q \\ &= \big(2(c-1)-\sqrt{Q}\big)^2 \\ 2(a-1)(c-1)(1-\cos\theta)\langle -1,d,a\rangle &= 4(a-1)^2-4(a-1)\sqrt{Q}+Q \\ &= \big(2(a-1)-\sqrt{Q}\big)^2 \end{aligned}$$

Since we have that $a \geq c$ then

$$\sqrt{Q} = \sqrt{2(a-1)(c-1)(1+\cos\theta)} \leq \sqrt{2(a-1)(a-1)2} = 2(a-1).$$

In particular one of the two following holds. Either $2(c-1) \geq \sqrt{Q}$ in which case

$$\sqrt{\langle -1,a,b\rangle}-\sqrt{\langle -1,b,c\rangle}+\sqrt{\langle -1,c,d\rangle}-\sqrt{\langle -1,d,a\rangle} = \frac{\big(2(a-1)+\sqrt{Q}\big)-\big(2(c-1)+\sqrt{Q}\big)+\big(2(c-1)-\sqrt{Q}\big)-\big(2(a-1)-\sqrt{Q}\big)}{\sqrt{2(a-1)(c-1)(1-\cos\theta)}} = 0,$$

or $2(c-1) \leq \sqrt{Q}$ in which case

$$\sqrt{\langle -1,a,b\rangle}-\sqrt{\langle -1,b,c\rangle}-\sqrt{\langle -1,c,d\rangle}-\sqrt{\langle -1,d,a\rangle} = \frac{\big(2(a-1)+\sqrt{Q}\big)-\big(2(c-1)+\sqrt{Q}\big)-\big(\sqrt{Q}-2(c-1)\big)-\big(2(a-1)-\sqrt{Q}\big)}{\sqrt{2(a-1)(c-1)(1-\cos\theta)}} = 0.$$

So for *every* arrangement of the four circles either

$$\sqrt{\langle -1, a, b\rangle} - \sqrt{\langle -1, b, c\rangle} + \sqrt{\langle -1, c, d\rangle} - \sqrt{\langle -1, d, a\rangle} = 0, \tag{3}$$

or

$$\sqrt{\langle -1, a, b\rangle} - \sqrt{\langle -1, b, c\rangle} - \sqrt{\langle -1, c, d\rangle} - \sqrt{\langle -1, d, a\rangle} = 0. \tag{4}$$

This concludes the proof of the statement of the theorem, but we still want to distinguish when the two cases hold. To do this we note that the only time that *both* of these can hold is if $\langle -1, p, q\rangle = 0$ for some appropriate pair of bends. This only occurs if we have $1/p + 1/q = 1$, i.e., the centers of the circles of bends -1, p and q are all collinear as shown in Figure 8.

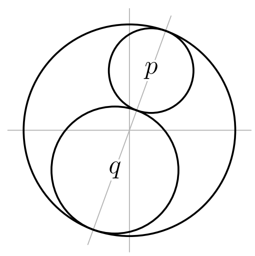
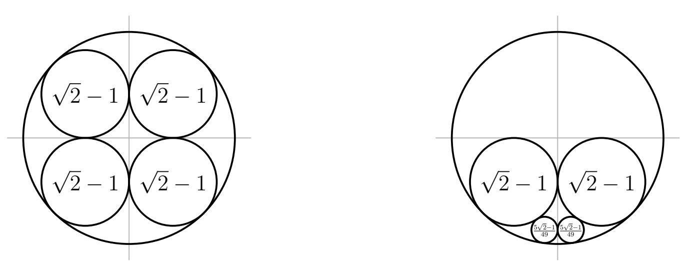


Figure 8: When $\langle -1, p, q\rangle = 0$.

If the center of the circle with bend -1 is inside of the quadrilateral formed by the centers of a, b, c, d then we can continuously deform the packing by rotation, deflation and inflation without ever having the center of the circle with bend -1 leave the interior of the quadrilateral to the configuration shown in Figure 9a. Similarly, if the center of the circle with bend -1 is outside the quadrilateral we can deform it continuously with the center remaining on the outside to the configuration shown in Figure 9b.



(a) Center on the inside. (b) Center on the outside.

Figure 9: Generic packings.

If we let a', b', c', d' denote the bends as it goes through the deformation then we note that these will all be continuous as will the quantities $\langle -1, a', b'\rangle$, $\langle -1, b', c'\rangle$, $\langle -1, c', d'\rangle$ and $\langle -1, d', a'\rangle$.

Suppose we start in the case where the center is on the inside and we deform to Figure 9a (where it is easy to check that (3) holds). If in the initial configuration we had (4) hold then at some point during the deformation we would have to simultaneously have (3) and (4) hold. But this implies we are in the situation shown in Figure 9a which shows that the center of the circle with bend -1 is not in the interior. A contradiction. So it must be if we start with the center on the inside that (3) holds. A similar argument shows if we start with the center of the circle with bend -1 on the outside that (4) holds.

In the case when all of the bends are positive we know that (3) or (4) holds and similar variational arguments allows us give similar conclusions about when each situation holds.

4. An application for G-packings

We have seen for flowers with four petals that there is a relationship among quantities related to the four interstices. One might hope that similar relationships hold for flowers with more petals, but this does not appear to be the case. For example, it is easy to check for the flower with six petals in Figure 10 that the alternating sum of the $\sqrt{\langle b_0, b_i, b_{i+1}\rangle}$ is not zero.

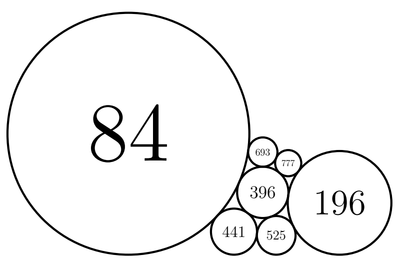


Figure 10: A flower with six petals.

Previously the authors [2] looked at packings (configurations of circles where the complement of the interiors consist of curvilinear triangles) and noticed that an important way to represent a packing was in the "standard" configuration wherein we have two circles of bend 0 which correspond to $y = \pm 1$ and two circles of bend 1 at the ends, one of which is centered at the origin, and the remaining circles in the middle. An example of a standard packing is shown in Figure 11.

It was also discovered that an important type of packing is one in which every circle is tangent to an even number of other circles. Such packings are known as Eulerian, since the tangency graph (i.e., the graph where each circle is a vertex

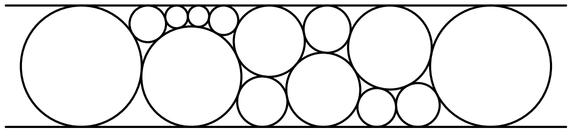


Figure 11: An example of a standard packing.

and we connect two vertices if the corresponding circles are tangent) are Eulerian graphs. For Eulerian packings we can two-color the interstices so no two adjacent interstices have the same color (this is a characterization of Eulerian packings). An example of this is shown in Figure 12 which gives the two-coloring of the standard packing shown in Figure 11.

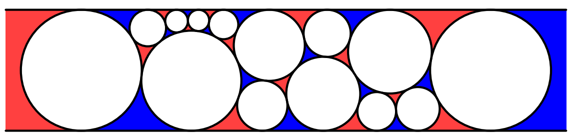


Figure 12: Two-coloring the interstices.

For some standard packings the sum of $\sqrt{\langle b_i, b_j, b_k\rangle}$ over all the red interstices is equal to the sum of $\sqrt{\langle b_i, b_j, b_k\rangle}$ over all the blue interstices. If this holds for *each* standard packing of a given configuration (there can be more than one but only finitely many for a given configuration) then the packing is called a G-packing[1]. It is possible for a configuration to have some of the packings to have the sum of the red and blue interstices equal while others do not, these are called quasi-G-packings, while if the sum is never equal in any standard packing then we call it an anti-G-packing (an example of an anti-G-packing is shown in Figure 13).

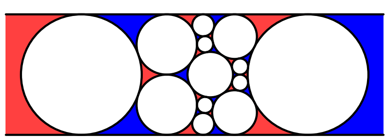


Figure 13: An example of an anti-G-packing.

We now give a sufficient condition for a packing to be a G-packing.

[1]The term G-packing is named after one of the authors (GG), who was the first one to discover Theorem 1 and the existence of such packings.

Theorem 2. *If the tangency graph of a packing is Eulerian, irreducible, and there is an independent covering set of vertices where each vertex in the covering set has degree four, then the corresponding packing is a G-packing.*

A tangency graph is irreducible if there is no triangle which is not a face. In terms of packings it means we cannot decompose it into smaller packings (see [2]).

Proof. Because the graph is irreducible we can now eliminate all the configurations shown in Figure 2 ((b) and (c) are trivial since there are no circles with negative bands in standard packings while (a) and (d) would give us a triangle which is not a face). In particular, for every circle surrounded by four circles we have that (1) holds.

By using the independent covering with vertices of degree four we can locally match up the sum of two red interstices with two blue interstices. This will catch each interstice exactly once and so we can conclude the sums are equal giving us the result. □

It is now an easy exercise to show that the packing in Figure 11 is a G-packing. Another example of a more complicated packing which is a G-packing is shown in Figure 14 (where we have marked the circles which correspond to the covering).

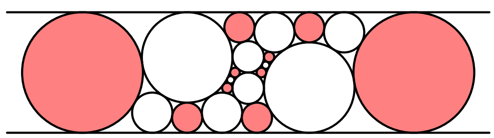


Figure 14: Another example of a G-packing with the covering marked.

While Theorem 2 gives a sufficient condition for a packing to be a G-packing, it is not necessary. For example, Theorem 2 only works when we have an *even* number of vertices, but there are G-packings with an odd number of vertices (some examples are given in the appendix).

We have the following count for the different types of packings.

n	Eulerian	G-packings	quasi-G-packings	anti-G-packings
10	2	2	0	0
11	2	1	1	0
12	8	4	3	1
13	8	0	8	0
14	32	14	13	5
15	57	0	47	10
16	185	41	89	55
17	466	6	270	190
18	1543	168	581	794

One thing to note is it appears that G-packings become rare as the number of circles increases. Another interesting thing to note is up through 18 circles there in only one packing which is a G-packing and not irreducible, even though this is not a requirement for a packing to be a G-packing, this packing is shown in Figure 15. (We have found five more reducible G-packing for 20 circles so as the number of circles increase it might happen that reducible G-packings are common.)

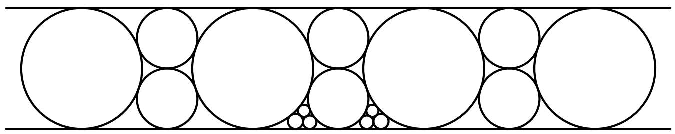
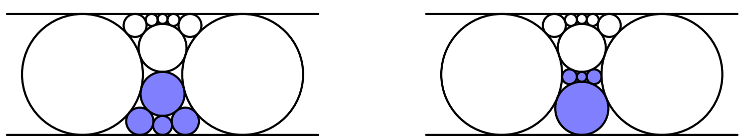


Figure 15: An example of a G-packing which is not irreducible.

Reflecting small regions in the packing

Examining the G-packings for at most 16 circles in the case when there are an even number of circles, the majority either follow from Theorem 2 or seem related in the sense there is a local switch that relates one to a G-packing for which the theorem can be used. To make this more concrete, consider the two G-packings shown in Figure 16.



(a) Theorem 2 doesn't apply. (b) Theorem 2 works.

Figure 16: Two G-packings.

Some circles have been highlighted and these circles are the only difference between the two packings. We can go from one packing to the other by locally "reflecting" the shaded circles. In general, for a four-cycle in the tangency graph with u and v opposite vertices on the cycle, we can take all the edges on the interior of the cycle connected to u and reconnect them to v and vice-versa. In the figure above the four-cycle consists of the four uncolored circles which surround the shaded circles, while the shaded circles correspond to the points in the interior of the four-cycle.

By adjusting the tangency graph we have to repack the configuration. However, it can be shown that in this special case of reflecting in the four-cycle that we only need to repack the circles on the interior of the four-cycle (i.e., all other circles can be packed as before). This explains why in Figure 16 the only difference was in the shaded circle and all other circles are unchanged.

Conjecture 1. *Reflecting the interior of a four-cycle inside of the tangency graph preserves the property of a packing being a G-packing.*

If we combine this with the results of Theorem 2 this seems to characterize most of the G-packings with an even number of vertices. It is still an open problem of how to characterize G-packings with an odd number of vertices, or even if there exists infinitely many of them.

5. Concluding remarks

Our motivation for Theorem 1 was driven by Apollonian circle packings (i.e., where the interiors are all disjoint). However, the proofs of Theorem 1 apply to more general situations where, for example, circles b_1 and b_3 might both have negative bend and intersect as shown in Figure 17. In this situation it can be shown that (1) holds.

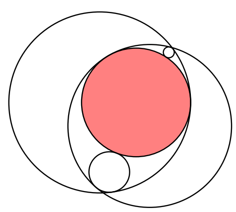


Figure 17: An example with intersecting circles.

The geometrical proof of Theorem 1 was to argue that there is a relationship among the bends of the orthogonal circles. A natural question is whether there is also a relationship among the circles that we can pack into the interstices (i.e., the circles formed by doing one stage of Apollonian circle packing). If we let b_{ij} denote the bend of the circle filling the interstice between the circle with bends b_0, b_i and b_j then for configurations shown in Figure 1 (i.e., where (1) holds), we have

$$b_{12} + b_{34} = b_{23} + b_{41}.$$

This follows from Descartes Circle Theorem which (among other things) implies that $b_{ij} = b_0 + b_i + b_j + 2\sqrt{\langle b_0, b_i, b_j\rangle}$ and Theorem 1 since

$$\begin{aligned} b_{12} + b_{34} &= \big(b_0 + b_1 + b_2 + 2\sqrt{\langle b_0, b_1, b_2\rangle}\big) + \big(b_0 + b_3 + b_4 + 2\sqrt{\langle b_0, b_3, b_4\rangle}\big) \\ &= \big(b_0 + b_2 + b_3 + 2\sqrt{\langle b_0, b_2, b_3\rangle}\big) + \big(b_0 + b_4 + b_1 + 2\sqrt{\langle b_0, b_4, b_1\rangle}\big) = b_{23} + b_{41}. \end{aligned}$$

One of our motivations for looking at G-packings is that many packings lying in simple fields (see [2]) are also G-packings. In both cases this might be a reflection of the role that symmetry can play in determining these two properties.

Also, when we looked at G-packings we restricted ourselves to standard packings. One advantage of this is that there are only finitely many standard packings for any Apollonian configuration so it is easy to test whether a given packing is a G-packing. Another advantage of this is that it allows for the existence of G-packings, i.e., if we make the requirement that the sum be the same over the red and blue interstices for any packing then no non-trivial packing seems to work. The problem is that we can continuously deform a G-packing where (1) holds for every flower with four petals to one where (1) holds for all but one flower with four petals which can destroy the balance of the sum over red and blue.

We believe that the relationships that we have discovered among the interstices to be the tip of a much larger iceberg for relationships in Apollonian circle packings that are waiting to be discovered.

Acknowledgments

Several of the illustrations used in this paper were created based on the algorithms given by Chuck Collins and Kenneth Stephenson [3, 9]. The software used to generate all of the Eulerian configurations was created by Gunnar Brinkmann and Brendan McKay [1]. We express our thanks for making publicly available the software and algorithms which aided immensely in the illustrating and writing of this note. The first author was supported by an NSF Postdoctoral fellowship.

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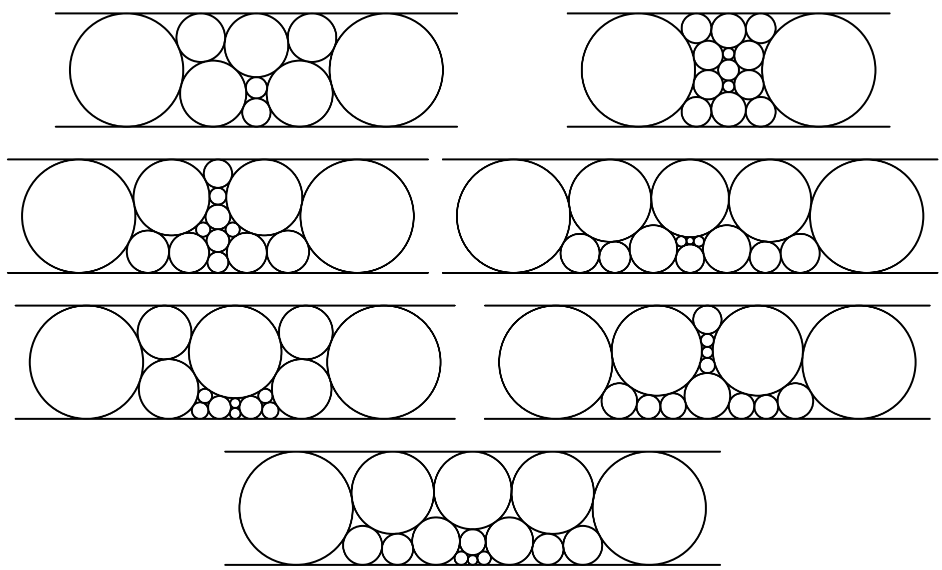
Appendix

The following are the known examples of G-packings with an odd number of circles.



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