# The Moebius transform of the triangular numbers

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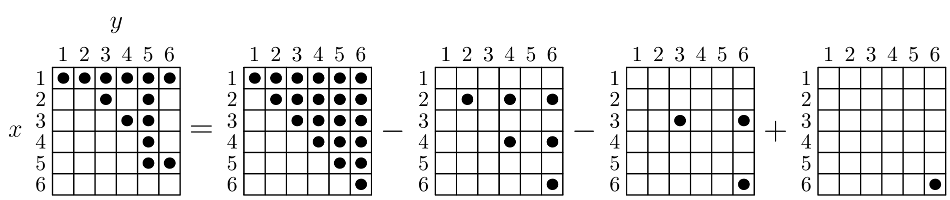
$$\mu(n) = \begin{cases} 1, & n = 1; \\ 0, & p^2 | n \text{ for some prime } p; \\ (-1)^r, & \begin{array}{l} n = p_1 p_2 \cdots p_r, \text{ where} \\ p_i \text{ are distinct primes} \end{array}. \end{cases}$$

$$T_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

**Theorem.** $\big|\{1 \le x \le y \le n : \gcd(x, y, n) = 1\}\big| = \displaystyle\sum_{d|n} \mu(n/d) T_d.$

Shown below is the case for $n = 6$, it is left to the reader to fill in the details.




$$\left|\left\{\begin{array}{c} x \le y \\ 1 = \gcd \end{array}\right\}\right| = \left|\left\{\begin{array}{c} x \le y \\ 1 \,|\, \gcd \end{array}\right\}\right| - \left|\left\{\begin{array}{c} x \le y \\ 2 \,|\, \gcd \end{array}\right\}\right| - \left|\left\{\begin{array}{c} x \le y \\ 3 \,|\, \gcd \end{array}\right\}\right| + \left|\left\{\begin{array}{c} x \le y \\ 6 \,|\, \gcd \end{array}\right\}\right|$$

More generally, a similar argument will show the following for all $k$ and $n$:

$$\big|\{1 \le x_1 \le x_2 \le \cdots \le x_k \le n : \gcd(x_1, x_2, \ldots, x_k, n) = 1\}\big| = \sum_{d|n} \mu(n/d) \binom{d+k-1}{k}.$$

$$b(n) = \begin{cases} 0, & n = 0; \\ \sum_{i=1}^{n} \mu(n), & n \ge 1. \end{cases} \qquad \text{(This is known as Merten's function.)}$$

**Corollary.** $\big|\{1 \le x_1 \le x_2 \le \cdots \le x_k \le n : \gcd(x_1, x_2, \ldots, x_k) = 1\}\big| = \displaystyle\sum_{i \ge 1} b(\lfloor n/i \rfloor) \binom{i+k-2}{k-1}.$

$k = 1$:

$$\sum_{i \ge 1} b(\lfloor n/i \rfloor) = \sum_{i \ge 1} \lfloor n/i \rfloor \mu(i) = \sum_{i=1}^{n} \sum_{d|i} \mu(d) = \sum_{i=1}^{n} [i = 1] = 1.$$

$k \ge 2$:

$$\big|\{1 \le x_1 \le x_2 \le \cdots \le x_k \le n : \gcd(x_1, x_2, \ldots, x_k) = 1\}\big| =$$

$$\sum_{m=1}^{n} \big|\{1 \le x_1 \le x_2 \le \cdots \le x_{k-1} \le m : \gcd(x_1, x_2, \ldots, x_{k-1}, m) = 1\}\big| =$$

$$\sum_{m=1}^{n} \sum_{d|m} \mu(m/d) \binom{d+k-2}{k-1} = \sum_{i \ge 1} \big(\mu(i/i) + \mu(2i/i) + \cdots + \mu(i\lfloor n/i \rfloor / i)\big) \binom{i+k-2}{k-1} = \sum_{i \ge 1} b(\lfloor n/i \rfloor) \binom{i+k-2}{k-1}.$$