Jumping Sequences

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Jumping patterns

A jump pattern is a decreasing sequence starting at n and going to 1, $(n = n_0, n_1, n_2, \ldots, 1 = n_k)$.

We will be considering jump patterns that arise from being cost minimizing with respect to some weight. Namely let $w(i, j)$ be the weight of jumping from j down to i then the weight of a jumping pattern is

$$w(n_1, n_0) + w(n_2, n_1) + \cdots + w(n_k, n_{k-1}) = \sum_{i=1}^{k} w(n_i, n_{i-1}).$$

Jumping sequences

For a fixed weight function we can encode cost minimizing jumping patterns in a jumping sequence.

Method to encode jumping sequence

Let $a(1) = 1$, $b(1) = 0$ and recursively for $n \geq 2$

$$\begin{aligned} b(n) &= \min_{1 \leq k \leq n-1} \big(w(k,n) + b(k)\big), \\ a(n) &= \text{lowest } k \text{ achieving the minimum for } b(n). \end{aligned}$$

The sequence $b(n)$ gives the minimal costs of jumping from n down to 1.

By choosing the "lowest" k we avoid possible ambiguity. For example, if $w(i,j) = j - i$ then *every* decreasing sequence has the same minimal cost.

Motivation

Places where jumping sequences have arisen:

- Key trees. Fix q, with $0 < q < 1$.

$$w(i,j) = \frac{1-q^j}{i} \qquad \text{and} \qquad w(i,j) = \frac{j}{i}.$$

- Analysis of Steiner trees.

$$w(i,j) = \frac{j^2}{i}.$$

Results for $w(i,j) = (1 - q^j)/i$

Theorem

Fix $0 < q < 1$, and let $(n = n_0, n_1, \ldots, n_k = 1)$ be a jump pattern along the reals. Then for $i = 2, \ldots, k$

$$\sqrt{2} \leq \frac{n_i}{n_{i-1}} \leq \frac{19}{4}.$$

Further if $n \leq 2$ or $q \leq e^{-1/e}$ then best pattern is $(n, 1)$; while if $n \geq 5$ and $q \geq 0.9$ then at least one intermediate jump should be taken.

Corollary

For n and q given the number of jumps needed is

$$\Theta\left(\min(\ln n, -\ln\left(\ln(\frac{1}{q})\right))\right).$$

Limiting case as $q \to 1$

As $q \to 1$ all the weights go to 0. This is not a very interesting case and so instead we look at "the derivative" to see which gives us the best result.

If we let $q = 1 - p$ for p small then binomial theorem gives $1 - q^x = px + O(p^2)$, or

$$\sum_{i=1}^{k} \frac{1 - q^{n_{i-1}}}{n_i} = p \sum_{i=1}^{k} \frac{n_{i-1}}{n_i} + O(p^2).$$

So we use $w(i,j) = j/i$ to handle the limiting case.

Jumping along the reals

If we relax the condition that we jump along integers and now allow to jump along reals then the situation is easy.

$$\begin{aligned} \frac{n_0}{n_1} + \frac{n_1}{n_2} + \cdots + \frac{n_{k-1}}{n_k} &\geq k \sqrt[k]{\frac{n_0}{n_1}\frac{n_1}{n_2}\cdots\frac{n_{k-1}}{n_k}} \\ &= k\sqrt[k]{n} \geq e \ln n. \end{aligned}$$

Theorem

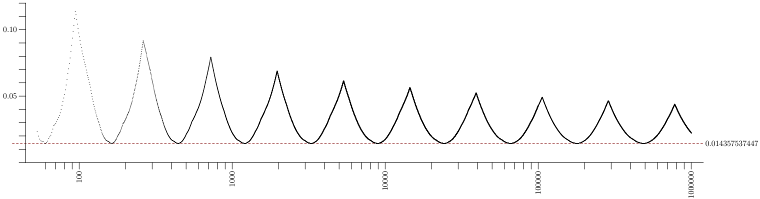
For the weight function $w(i,j) = j/i$ where we take jumps along the real numbers the minimal weight of jumping from n down to 1 is

$$b(n) = e \ln n + O\left(\frac{1}{\ln n}\right).$$

Further the optimal jump pattern is a geometric sequence with ratio $e^{1+o(1)}$.

Jumping along the integers

If we jump along the integers then we are more restricted in how we can jump. How much will this increase the cost?



$$\text{Error} = b(n) - e \ln n$$

Bottoms of the troughs are located at $(1, 3, 8, 22, 60, 163, 443, 1204, 3273, 8897, 24185, 65742, \ldots)$.

An important sequence

$c = (1, 3, 8, 22, 60, 163, 443, 1204, 3273, 8897, 24185, \ldots)$

$c(n)$ is the sequence defined by

$$\begin{aligned} c(0) &= 1, \\ c(n+1) &= \lfloor e \cdot c(n) + 0.5 \rfloor, \qquad \text{for } n \geq 0. \end{aligned}$$

(Multiply by e then round at each step.)

Properties of this sequence:

- $c(n) \approx (1.09800209936683\ldots)e^n$
- $$\lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{c(i)}{c(i-1)} - e \ln\left(c(n)\right) \right) = 0.014357537447\ldots = \alpha$$

Theorem

For the weight function $w(i,j) = j/i$ where we take jumps along the integers the minimal weight from n down to 1 satisfies

$$b(n) \leq e \ln n + (0.014357537\ldots) + O(\frac{1}{\ln n}).$$

Sketch of proof

For large n let $\ell = \ln \ln n$. Now form a sequence (starting at 1) by using $c(0), c(1), \ldots, c(\ell)$, for the remainder we find the optimal jump sequence along the reals between $c(\ell)$ and n, call these $\delta(0) = c(\ell), \delta(1), \ldots, \delta(k) = n$ then round each term to the nearest integer for sequence by $d(0), d(1), \ldots, d(k)$.

$$\text{Cost} \leq \underbrace{e \ln c(\ell) + \alpha}_{\text{cost of } c\text{'s}} + \underbrace{e \ln \frac{n}{c(\ell)} + O(\frac{1}{\ln \frac{n}{c(\ell)}})}_{\text{cost of } \delta\text{'s}} + \underbrace{O(\frac{1}{c(\ell)})}_{\leq \sum |\delta(n) - d(n)|}$$

How good is the approximation?

For what n is $b(n) < e \ln n + 0.014357537447\ldots$?

By definition of α, each $c(i)$ satisfies this relationship.

The first term which is not $c(i)$ and satisfies this relationship is $c(212) + 1$, or

$$1291318384051937580859421657451339543637942426407958928241564048452309868616554970528509821 85$$

With the exception of the first ≈ 200 terms from the sequence c

$$b(n) > e \ln N + (0.014357537447\ldots) - \frac{1}{10^{180}}.$$

Looking more closely at the sequence

For the weight function $w(i,j) = j/i$ the jump sequence starts

$$a = (1,1,1,1,2,2,3,3,3,3,3,5,5,5,7,7,7,7,8,8,8,8,8,8,9,\ldots)$$

Let us compress the sequence and list only the numbers which appear.

$$\mathcal{A}_1 = \{1,2,3,5,7,8,9,10,13,16,17,18,19,20,21,22,23,25,\ldots\}$$

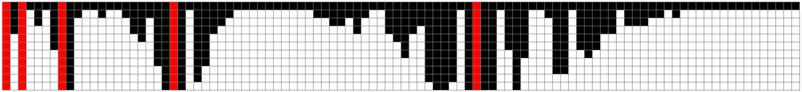
Alternatively, $\mathcal{A}_1$ are the terms which show up as the first term in some jump sequence. More generally we can look at $\mathcal{A}_k$ which are the terms which show up as the kth term in some jump sequence.

The first few $\mathcal{A}_k$ for $w(i,j) = j/i$

$$\mathcal{A}_0=\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,\ldots\}$$
$$\mathcal{A}_1=\{1,2,3,5,7,8,9,10,13,16,17,18,19,20,21,22,23,25,\ldots\}$$
$$\mathcal{A}_2=\{1,2,3,5,7,8,9,17,18,20,21,22,23,25,26,27,28,42,\ldots\}$$
$$\mathcal{A}_3=\{1,2,3,7,8,9,17,18,20,21,22,23,25,26,27,45,49,\ldots\}$$
$$\mathcal{A}_4=\{1,3,7,8,9,18,20,21,22,23,25,26,50,51,54,55,56,\ldots\}$$
$$\mathcal{A}_5=\{1,3,7,8,9,20,21,22,23,25,26,51,54,55,56,57,59,\ldots\}$$
$$\mathcal{A}_6=\{1,3,8,9,20,21,22,23,25,26,51,54,55,56,57,59,60,\ldots\}$$
$$\mathcal{A}_7=\{1,3,8,9,20,21,22,23,25,26,54,55,56,57,59,60,61,\ldots\}$$
$$\mathcal{A}_8=\{1,3,8,9,21,22,23,25,54,55,56,57,59,60,61,62,65,\ldots\}$$
$$\mathcal{A}_9=\{1,3,8,9,21,22,23,25,54,55,56,57,59,60,61,62,65,\ldots\}$$
$$\mathcal{A}_{10}=\{1,3,8,9,21,22,23,55,56,59,60,61,62,65,144,145,\ldots\}$$

More on sieving

Graphically we get the following picture for the sieving process.



These sets satisfy $\mathcal{A}_{k+1} \subseteq \mathcal{A}_k$. Now let

$$\mathcal{A}_\infty = \bigcap_{k \geq 0} \mathcal{A}_k.$$

(These correspond to terms which can show up arbitrarily deep in some jump pattern.)

The set $\mathcal{A}_\infty$ exists for every weight function.

Sieving for the set $w(i,j) = j/i$

Conjecture

For $w(i,j) = j/i$

$$\mathcal{A}_\infty = \{1, 3, 8, 22, 60, 163, 443, 1204, 3273, 8897, \ldots\}$$

Observation

If $b(n) > e \ln n + (0.014357537\ldots)$ then $n \notin \mathcal{A}_\infty$.

This can be used to show that the first 211 terms of $\mathcal{A}_\infty$ come from the sequence c. (Recall that this sequence is the one that starts with 1 and then multiply by e, round, repeat.)

A different weight function

We can look at other weight functions. Consider $w(i,j) = (i+j)/i^2$.

$$\mathcal{A}_0 = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,\ldots\}$$
$$\mathcal{A}_1 = \{1,2,3,5,7,11,12,13,24,26,27,28,29,30,31,32,34,\ldots\}$$
$$\mathcal{A}_2 = \{1,2,3,5,11,12,13,27,28,29,30,64,65,67,68,69,70,\ldots\}$$
$$\mathcal{A}_3 = \{1,2,5,12,28,29,69,70,71,163,164,166,167,168,169,\ldots\}$$
$$\mathcal{A}_4 = \{1,2,5,12,29,69,70,168,169,170,402,403,405,406,\ldots\}$$
$$\mathcal{A}_5 = \{1,2,5,12,29,70,168,169,407,408,409,979,980,982,\ldots\}$$
$$\mathcal{A}_6 = \{1,2,5,12,29,70,169,407,408,984,985,986,2372,\ldots\}$$
$$\mathcal{A}_7 = \{1,2,5,12,29,70,169,408,984,985,2377,2378,\ldots\}$$
$$\mathcal{A}_8 = \{1,2,5,12,29,70,169,408,985,\ldots\}$$

The Pell numbers

The sequence numbers $1, 2, 5, 12, 29, 70, 169, 408, 985$ form the start of the Pell numbers. These are given by

$$p_1 = 1, \quad p_2 = 2 \quad \text{and} \quad p_n = 2p_{n-1} + p_{n-2}.$$

From this it follows that

$$p_n = \frac{1}{2\sqrt{2}}\left(r^n - \bar{r}^n\right), \qquad \text{where } r = 1 + \sqrt{2}, \bar{r} = 1 - \sqrt{2}.$$

Alternatively, these can be generated by

$$p_1 = 1 \quad \text{and} \quad p_{n+1} = \lfloor r \cdot p_n + 0.5 \rfloor, \quad \text{for } n \geq 0.$$

(Same multiply, round, repeat pattern as we saw before.)

Sieving to the Pell numbers

Theorem

For the weight function $w(i,j) = (i+j)/i^2$

$$\mathcal{A}_\infty = \{1, 2, 5, 12, 29, 70, 169, 408, 985, \ldots\},$$

the Pell numbers.

To establish the result we will show by induction for every $k \geq 1$ that as n gets large that the jumping sequence (starting at 1 and going to n) must begin $1 = p_1, p_2, \ldots, p_k$.

The base case is trivial since we always are starting with 1.

Jumping along the Pell numbers

Let

$$W = \sum_{k=1}^{\infty} \frac{p_k + p_{k+1}}{p_k^2} = 5.91570247\ldots.$$

(This is the weight of the infinite jump sequence starting at 1 and going to ∞ by jumping along the Pell numbers.)

Observation

The sequence $b(n)$ is increasing, so

$$b(n) \le b(p_n) \le \sum_{k=1}^{n-1} \frac{p_k + p_{k+1}}{p_k^2} < W.$$

(It will follow from the previous theorem that $b(n) \to W$ as $n \to \infty$.)

Approximation result

We will need to control the size of the tail of W.

Lemma

We have

$$\sum_{i=2j+1}^{\infty} \frac{p_i + p_{i+1}}{p_i^2} < \frac{2(2+\sqrt{2})}{r^{2j}} + \frac{1}{8r^{6j}}.$$

Lemma

For any real numbers a_i,

$$\lim_{n\to\infty} \left(\inf_{1=a_0\leq a_1\leq a_2 \cdots \leq a_n} \sum_{i=1}^{n} \frac{a_{i-1}+a_i}{a_{i-1}^2} \right) = 3 + 2\sqrt{2}.$$

Sketch of proof (part 1)

We already have the base case. Suppose that we know that for n large enough we must start $p_1, p_2, \ldots, p_k$. Suppose that t is the next term for arbitrarily large n, i.e., we have a jump pattern of the form $(\beta_\ell, \cdots, \beta_1, t, p_k, \ldots, p_1)$.

$$W > \sum_{i=1}^{k-1} \frac{p_i + p_{i+1}}{p_i^2} + \frac{p_k + t}{p_k^2} + \frac{1}{t} \sum_{i=0}^{\ell-1} \frac{(\beta_i/t) + (\beta_{i+1}/t)}{(\beta_i/t)^2}$$

Since this holds for arbitrarily large ℓ, it must also hold for the limit, so we have

$$W \geq \sum_{i=1}^{k-1} \frac{p_i + p_{i+1}}{p_i^2} + \frac{p_k + t}{p_k^2} + \frac{3 + 2\sqrt{2}}{t}$$

Sketch of proof (part 2)

Substituting the definition of W and simplifying we must have

$$\sum_{i=k+1}^{\infty} \frac{p_i + p_{i+1}}{p_i^2} \geq \frac{t - p_{k+1}}{p_k^2} + \frac{3 + 2\sqrt{2}}{t}. \qquad (*)$$

By our definitions, $t = p_{k+1}$ satisfies $(*)$.

Observation

$$f(t) = \frac{t - p_{k+1}}{p_k^2} + \frac{3 + 2\sqrt{2}}{t}$$

is decreasing for $t \leq p_{k+1} - 1$ and increasing for $t \geq p_{k+1} + 1$.

Finally, we check that $(*)$ does not hold for $t = p_{k+1} \pm 1$.

Yet one more weight function

Theorem

For $w(i,j) = (ij + j^2)/i^3$,

$$\mathcal{A}_\infty = \{1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots\},$$

the Fibonacci numbers.

Note that the Fibonacci numbers can be defined by $F(0) = 1$ and then multiply by $\phi = (1 + \sqrt{5})/2$, round, repeat.

What other interesting sequences can we get as $\mathcal{A}_\infty$ for some weight function?

Not just multiply, round, repeat.

For the weight function $w(i,j) = (i^2 + ij + j^2)/i^3$, we should expect to use ratio $q = 1.73990787\ldots$ (root of $2q^3 - 2q^2 - 2q - 1 = 0$).

$$\begin{aligned} \text{expected } \mathcal{A}_\infty : &\quad \{1, 2, 3, 5, 9, 16, 28, 49, \ldots\} \\ \text{actual } \mathcal{A}_\infty : &\quad \{1, 2, 4, 7, 12, 21, 37, 64, \ldots\} \end{aligned}$$

What happened is that instead of 3 (as we would expect rounding $3.47981574\ldots$) we end up with 4.

Still more to be done!

THANK YOU