Cospectral graphs for both the adjacency and normalized Laplacian matrices

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Abstract

In this note we show how to construct two distinct bipartite graphs which are cospectral for both the adjacency and normalized Laplacian matrices by "unfolding" a base bipartite graph in two different ways.

1 Introduction

Spectral graph theory uses the eigenvalues of matrices associated with a graph to determine structural properties of that graph. This approach can have limitations however since two distinct graphs can share the same eigenvalues (see [3] for a discussion about graphs determined by their spectrum).

There have been several constructions given which produce cospectral graphs (see [4, 5]). However, these constructions have focused on the adjacency matrix, the (combinatorial) Laplacian and the signless (combinatorial) Laplacian. In this note we will give a construction for a large class of graphs which are cospectral with respect to the normalized Laplacian (popularized by the work of Chung [2]). Our method of construction is a special case of a more general result given in [1], we will include simplified proofs here for completeness.

These graphs will also be cospectral with respect to the adjacency matrix and thus give examples of non-regular graphs which are simultaneously cospectral with respect to both the adjacency and the normalized Laplacian. Examples were previously known of regular graphs which are cospectral with respect to the adjacency matrix and so also trivially cospectral with respect to the normalized Laplacian.

Our construction follows from a result on matrices. For a matrix M, let $\sigma(M)$ be the (multi-)set of eigenvalues of M, i.e., the spectrum of M. Further, for a nonnegative symmetric matrix M with positive row (and hence column) sums let $\mathcal{L}(M) = I - D^{-1/2}MD^{-1/2}$ where D is the diagonal matrix with diagonal entries composed of the row (or column) sums of M.

Lemma 1. *Let B be a $p{\times}q$ matrix. Then*

$$\begin{aligned}
\sigma\left(\begin{bmatrix} O & B & B \\ B^* & O & O \\ B^* & O & O \end{bmatrix}\right) &= \sqrt{2}\,\sigma\left(\begin{bmatrix} O & B \\ B^* & O \end{bmatrix}\right) \cup \{\underbrace{0,0,\ldots,0}_{\times q}\}, \quad \textit{and} \\
\sigma\left(\begin{bmatrix} O & B & O \\ B^* & O & B^* \\ O & B & O \end{bmatrix}\right) &= \sqrt{2}\,\sigma\left(\begin{bmatrix} O & B \\ B^* & O \end{bmatrix}\right) \cup \{\underbrace{0,0,\ldots,0}_{\times p}\}.
\end{aligned}$$

Further, if B is nonnegative and has positive row and column sums then

$$\begin{aligned}
\sigma\left(\mathcal{L}\left(\begin{bmatrix} O & B & B \\ B^* & O & O \\ B^* & O & O \end{bmatrix}\right)\right) &= \sigma\left(\mathcal{L}\left(\begin{bmatrix} O & B \\ B^* & O \end{bmatrix}\right)\right) \cup \{\underbrace{1,1,\ldots,1}_{\times q}\}, \quad \textit{and} \\
\sigma\left(\mathcal{L}\left(\begin{bmatrix} O & B & O \\ B^* & O & B^* \\ O & B & O \end{bmatrix}\right)\right) &= \sigma\left(\mathcal{L}\left(\begin{bmatrix} O & B \\ B^* & O \end{bmatrix}\right)\right) \cup \{\underbrace{1,1,\ldots,1}_{\times p}\}.
\end{aligned}$$

Proof. We will prove the first statement in both cases (the second statements are handled similarly). Let $\left[\begin{smallmatrix}\mathbf{x}\\ \mathbf{y}\end{smallmatrix}\right]$ be an eigenvector for the eigenvalue λ and the matrix $\left[\begin{smallmatrix} O & B \\ B^* & O\end{smallmatrix}\right]$ (i.e., so $B\mathbf{y} = \lambda\mathbf{x}$ and $B^*\mathbf{x} = \lambda\mathbf{y}$). Then consider the following,

$$\begin{bmatrix} O & B & B \\ B^* & O & O \\ B^* & O & O \end{bmatrix}\begin{bmatrix} \sqrt{2}\,\mathbf{x} \\ \mathbf{y} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} 2B\mathbf{y} \\ \sqrt{2}\,B^*\mathbf{x} \\ \sqrt{2}\,B^*\mathbf{x} \end{bmatrix} = \sqrt{2}\lambda\begin{bmatrix} \sqrt{2}\,\mathbf{x} \\ \mathbf{y} \\ \mathbf{y} \end{bmatrix},$$

so that $\sqrt{2}\,\lambda$ is an eigenvalue for the enlarged matrix. For the remaining eigenvalues let $\mathbf{e}_i$ denote the vector of length q which is 1 in the ith slot and 0 otherwise. Then

$$\begin{bmatrix} O & B & B \\ B^* & O & O \\ B^* & O & O \end{bmatrix}\begin{bmatrix} \mathbf{0} \\ \mathbf{e}_i \\ -\mathbf{e}_i \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{1}$$

since these vectors are orthogonal to the ones previously given it shows that the remaining q eigenvalues are 0.

For the second part we first note that $\hat{\mathbf{x}}$ is an eigenvector of $\mathcal{L}(M)$ if and only if $\hat{\mathbf{y}} = R^{-1/2}\hat{\mathbf{x}}$ (known as the harmonic eigenvector) satisfies $(R - M)\hat{\mathbf{y}} = \lambda R\hat{\mathbf{y}}$. So for an eigenvalue λ let $\left[\begin{smallmatrix}\mathbf{x}\\ \mathbf{y}\end{smallmatrix}\right]$ be chosen to satisfy

$$\begin{bmatrix} R_1 & -B \\ -B^* & R_2 \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} R_1\mathbf{x} - B\mathbf{y} \\ -B^*\mathbf{x} + R_2\mathbf{y} \end{bmatrix} = \lambda\begin{bmatrix} R_1\mathbf{x} \\ R_2\mathbf{y} \end{bmatrix}$$

(where R_1 and R_2 are the partitioned parts of R). Then consider the following

$$\begin{bmatrix} 2R_1 & -B & -B \\ -B^* & R_2 & O \\ -B^* & O & R_2 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} 2R_1\mathbf{x} - 2B\mathbf{y} \\ -B^*\mathbf{x} + R_2\mathbf{y} \\ -B^*\mathbf{x} + R_2\mathbf{y} \end{bmatrix} = \lambda \begin{bmatrix} 2R_1\mathbf{x} \\ R_2\mathbf{y} \\ R_2\mathbf{y} \end{bmatrix},$$

so that λ is also an eigenvalue of the enlarged matrix. For the remaining eigenvalues we can use the same eigenvector and relationship given in (1), in particular since these vectors, $\hat{\mathbf{y}}$, satisfy $M\hat{\mathbf{y}} = \mathbf{0}$ then it follows that $(R - M)\hat{\mathbf{y}} = R\hat{\mathbf{y}}$ showing that these are q additional eigenvectors associated with an eigenvalue of 1, which completes the spectrum. □

2 The construction

Let G be a bipartite graph with vertex set $V(G) = V_1 \cup V_2$, and an edge set $E(G)$ where edges go between V_1 and V_2. Then we construct new graphs G_1 and G_2 where

$$\begin{aligned} V(G_1) &= V_1 \cup V_2 \cup V_2', \\ E(G_1) &= \big\{\{v_1(i), v_2(j)\}, \{v_1(i), v_2'(j)\} \,\big|\, \{v_1(i), v_2(j)\} \in E(G)\big\}; \\ V(G_2) &= V_1 \cup V_1' \cup V_2, \\ E(G_1) &= \big\{\{v_1(i), v_2(j)\}, \{v_1'(i), v_2(j)\} \,\big|\, \{v_1(i), v_2(j)\} \in E(G)\big\}. \end{aligned}$$

Intuitively this construction can be thought of as taking a bipartite graph and "unfolding" it either along V_1 or V_2 to construct the new graphs. An example of this construction is shown in Figure 1.

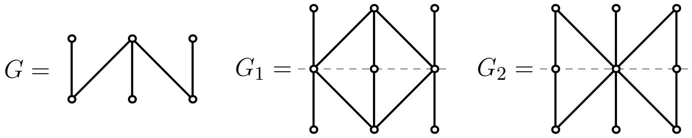

Figure 1: Examples of constructing G_1 and G_2.

Theorem 2. *Let G be a given bipartite graph with $|V_1| = p \le |V_2| = q$, and $A(G)$ denote the adjacency matrix of G. Then $\sigma\big(A(G_1)\big)$ and $\sigma\big(A(G_2)\big)$ differ by $(q - p)$ eigenvalues of 0. If further, G has no isolated vertices and $\mathcal{L}(G)$ denotes the normalized Laplacian matrix of G, then $\sigma\big(\mathcal{L}(G_1)\big)$ and $\sigma\big(\mathcal{L}(G_2)\big)$ differ by $(q - p)$ eigenvalues of 1.*

In particular, if G is a bipartite graph with $|V_1| = |V_2|$ and no isolated vertices then G_1 and G_2 are cospectral both with respect to the adjacency matrix and the normalized Laplacian.

Proof. The statement follows by noting that the form of the matrices $A(G)$, $A(G_1)$ and $A(G_2)$ are those in the statement of Lemma 1, and that the normalized Laplacian matrix of a graph is $\mathcal{L}(A(G))$. $\square$

We now have a construction for cospectral graphs, it remains to find examples where these graphs are non-isomorphic. It is easy to check that when the maximum degree of vertices in V_1 is not equal to the maximum degree of vertices in V_2, that the corresponding graphs G_1 and G_2 have different maximum degrees and hence are non-isomorphic. An example of this situation is shown in Figure 1.

In the case when $p < q$ the resulting graphs G_1 and G_2 can be made cospectral with respect to the adjacency matrix by the addition of isolated vertices to the smaller graph (there is not a similar operation for the normalized Laplacian). For instance if we start with the path on three vertices then G_1 and G_2 are a four-cycle and a star on five vertices, by adding an isolated vertex to the four-cycle the two graphs are now cospectral and give the famous Saltire pair (see [5]).

There still remains many problems in this direction to be worked on. For instance, is there an example of two non-regular graphs which are cospectral with respect to the adjacency, combinatorial Laplacian and normalized Laplacian at the same time? (The graphs shown in Figure 1 have different numbers of spanning trees and so do not give an example.) Further, all our examples using this construction will be bipartite graphs, are there more general constructions which can be used to make cospectral graphs with respect to the normalized Laplacian which have arbitrarily high chromatic number?

References

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