Tangent Line Transformations

Steven Butler

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This paper presents a transformation of a curve to a curve, based on the curve's tangent lines, that has a surprising property. Besides its intrinsic interest, it can be used in calculus classes, as a source of exercises or a starting point for further exploration.

Given the parametric representation of a curve, $(x(t), y(t))$, the equation of its tangent line at t is

$$y - y(t) = \frac{y'(t)}{x'(t)}(x - x(t)),$$

with slope

$$\frac{y'(t)}{x'(t)}$$

and y-intercept

$$y(t) - \frac{y'(t)}{x'(t)}x(t).$$

Let us define a transformed curve, $(x^*(t), y^*(t))$, by putting

$$x^*(t) = \frac{y'(t)}{x'(t)} \quad \text{and} \quad y^*(t) = y(t) - \frac{y'(t)}{x'(t)}x(t).$$

The new curve contains all of the information about the original curve's tangent lines. For example, it crosses the y-axis whenever the original curve has a horizontal tangent line.

The straight line $y = mx + b$, or $(x(t), y(t)) = (t, mt + b)$, has only one tangent line and thus a trivial transformation to a point, $(x^*(t), y^*(t)) = (m, b)$.

Some conic sections transform to conic sections. For example, the image of the unit circle $(x(t), y(t)) = (\sin t, \cos t)$ is

$$(x^*(t), y^*(t)) = \left(\frac{-\sin t}{\cos t}, \cos t + \frac{\sin t}{\cos t} \cdot \sin t\right) = \left(-\frac{\sin t}{\cos t}, \frac{1}{\cos t}\right),$$

the hyperbola $(x^*)^2 + 1 = (y^*)^2$. Similarly, ellipses with center at the origin and axes parallel to the coordinate axes transform to hyperbolas. Students could investigate the relation between the original curve's foci, directrices, and eccentricity and those of its transform, and they could investigate the transformations of tilted ellipses.

The image of the parabola $y = ax^2 + bx + c$ is another parabola. If

$$(x(t), y(t)) = (t, at^2 + bt + c) \quad \text{then} \quad (x^*(t), y^*t)) = (2at + b, -at^2 + c)\,.$$

It is an exercise in algebra to show that the new curve is

$$y^* = A(x^*)^2 + Bx^* + C \quad \text{with} \quad A = -\frac{1}{4a},\ B = \frac{b}{2a},\ C = -\frac{b^2}{4a} + c\,.$$

For one more example, the image of $y = e^x$ is $y^* = x^*(1 - \ln x^*)$. Readers can investigate other curves.

The surprising property of the transformation is that when it is applied twice, the original curve reappears, reflected about the y-axis. For example, the transformed parabola

$$\begin{aligned}(2at + b, -at^2 + c) &\to \left(\frac{-2at}{2a}, -at^2 + c - \left(\frac{-2at}{2a}\right)(2at + b)\right) \\ &= (-t, at^2 + bt + c).\end{aligned}$$

To show that this happens in general is a good exercise in differentiation and algebra.

Readers might be interested in [**1**], which was a motivation for this paper.

Reference

1. A. Horwitz, Reconstructing a function from its set of tangent lines, *American Mathematical Monthly* **96** (9) (1989) 807–813.

Cheap Stadium

Marc Brodie (College of St. Benedict, mbrodie@csbsju.edu), who reads newspapers critically, noticed the following in the Minneapolis *Star Tribune*, over a color picture of a football field:

> DID YOU KNOW? The Patriots' new home, Gillette Stadium, cost approximately \$325 to build. It replaces Foxboro Stadium, which cost approximately \$6 million.

The Shadow, an old radio character, had "the power to cloud men's minds so they could not see him." Numbers can cloud some people's minds as well. At least the cost wasn't given as "\$325 dollars."