DETERMINING THE UNDERLYING FUNCTIONS FOR THE CAUCHY POWER AND EXPONENTIAL FORMS

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Introduction

There are four basic Cauchy functional forms, namely:

$$\begin{aligned} u(x,y) &= f(x+y) - f(x) - f(y) && \text{Linear,}\\ u(x,y) &= f(xy) - f(x) - f(y) && \text{Logarithmic,}\\ u(x,y) &= f(x+y) - f(x)f(y) && \text{Exponential,}\\ u(x,y) &= f(xy) - f(x)f(y) && \text{Power.} \end{aligned}$$

The forms derive their names from the *underlying functions* (i.e. the $f(\cdot)$) that solves the homogeneous case when $u(x,y) = 0$.

With the underlying function in hand it is easy to derive the function $u(x,y)$. In this note we will go backwards by starting with a given $u(x,y)$ and then derive the underlying function, if it exists, for the exponential and power forms.

1. Finding the underlying functions

To find the underlying functions we will start with the basic relationship the form satisfies and try to manipulate it so that we can isolate f and thus find the function. This is shown in the following proof.

Theorem 1. *Given a function $u(x,y)$ defined on $\mathbb{R}^2$ which has a Cauchy exponential form and there exists at least one point such that $u(a,b) \neq 0$ then the underlying function satisfies the following:*

$$f(x) = \frac{u(x+a,b) - u(x,a+b)}{u(a,b)} + Ku(x,a)$$

Where K is a constant that satisfies the following quadratic equation.

$$K^2 - K\frac{u(2b,a)}{[u(a,b)]^2} + \frac{u(b,b)u(a,b) + u(2b,a+b) - u(2b+a,b)}{[u(a,b)]^3} = 0$$

Proof. For our proof we will use two variations of the exponential form, namely:

$$u(x,y) = f(x+y) - f(x)f(y)$$
$$f(x+y) = u(x,y) + f(x)f(y)$$

The approach is based on research into the linear Cauchy form done with Donald R. Snow of Brigham Young University.

Using these we get the following:

$$\begin{aligned} u(x+a,b) &= f((x+a)+b) - f(x+a)f(b) \\ &= f(x+(a+b)) - f(x+a)f(b) \\ &= [u(x,a+b) + f(x)f(a+b)] - f(x+a)f(b) \\ &= u(x,a+b) + f(x)[u(a,b)+f(a)f(b)] - \\ &\qquad [u(x,a)+f(x)(f(a)]f(b) \\ &= u(x,a+b) + f(x)u(a,b) - f(b)u(x,a) \end{aligned}$$

Since $u(a,b) \neq 0$ we can rearrange and solve for $f(x)$. Doing that gives us the following:

$$\begin{aligned} f(x) &= \frac{u(x+a,b)-u(x,a+b)}{u(a,b)} + \frac{f(b)}{u(a,b)}u(x,a) \\ &= \frac{u(x+a,b)-u(x,a+b)}{u(a,b)} + Ku(x,a) \qquad \text{where } K = \frac{f(b)}{u(a,b)}. \end{aligned}$$

Note here that K is a constant, but not an arbitrary one. To solve for K we will plug in a point and simplify. Since our function is an exponential form, the function will be symmetric, that is $u(x,y) = u(y,x)$. Using this fact, we get the following:

$$\begin{aligned} u(b,b) &= f(2b) - [f(b)]^2 \\ &= \frac{u(2b+a,b)-u(2b,a+b)}{u(a,b)} + Ku(2b,a) - \\ &\qquad \left[\frac{u(b+a,b)-u(b,a+b)}{u(a,b)} + Ku(b,a)\right]^2 \\ &= \frac{u(2b+a,b)-u(2b,a+b)}{u(a,b)} + Ku(2b,a) - K^2[u(a,b)]^2 \end{aligned}$$

Therefore K is a root of the quadratic equation

$$K^2 - K\frac{u(2b,a)}{[u(a,b)]^2} + \frac{u(b,b)u(a,b)+u(2b,a+b)-u(2b+a,b)}{[u(a,b)]^3} = 0.$$

□

Theorem 2. *Given a function $u(x,y)$ defined on $\mathbb{R}^2$ which has a Cauchy power form and there exists at least one point such that $u(a,b) \neq 0$ then the underlying function satisfies the following:*

$$f(x) = \frac{u(ax,b)-u(x,ab)}{u(a,b)} + Ku(x,a)$$

Where K is a constant that satisfies the following quadratic equation.

$$K^2 - K\frac{u(b^2,a)}{[u(a,b)]^2} + \frac{u(b,b)u(a,b)+u(b^2,ab)-u(ab^2,b)}{[u(a,b)]^3} = 0$$

Proof. The proof is identical to that for the exponential form. The only change is to replace all of the additions inside the arguments by multiplications. □

These theorems show how to generate an underlying function. They can also be used to test whether or not a given function $u(x, y)$ is of the power or exponential form by noting that for them to be of the form there must be an underlying function. If no underlying function exists, then it must be that they are not of that form. Therefore, to test whether or not a given function is of the correct form we assume it is and construct what the underlying function should look like and see if there is any constant K such that the underlying function will produce the original. If no such constant exists, then the function is not of the power or exponential form.

2. An exponential form example

Given $u(x, y) = d\sin(x)\sin(y)$ where $d \neq 0$ we will attempt to find the underlying function for the function if it were an exponential form.

We will start by assuming that it is of the exponential form and then we can apply theorem 1 since $u(\frac{\pi}{2}, \frac{\pi}{2}) = d \neq 0$. This will give us the following result:

$$\begin{aligned} f(x) &= \frac{d\sin(x+\frac{\pi}{2})\sin(\frac{\pi}{2}) - d\sin(x)\sin(\frac{\pi}{2}+\frac{\pi}{2})}{d\sin(\frac{\pi}{2})\sin(\frac{\pi}{2})} + Kd\sin(x)\sin(\frac{\pi}{2}) \\ &= \cos(x) + dK\sin(x) \end{aligned}$$

All that remains is to determine if there exists a value of K that will make the underlying function generate the original function. We will do this by comparing the given $u(x, y)$ to what the underlying function would give and then solve for K. This process is shown below.

$$\begin{aligned} d\sin(x)\sin(y) &= u(x, y) \\ &= f(x+y) - f(x)f(y) \\ &= [\cos(x+y) + dK\sin(x+y)] - \\ &\quad\ [\cos(x) + dK\sin(x)][\cos(y) + dK\sin(y)] \\ &= \cos(x+y) - \cos(x)\cos(y) + \\ &\quad\ dK[\sin(x+y) - \cos(x)\sin(y) - \cos(y)\sin(x)] - \\ &\quad\ d^2K^2\sin(x)\sin(y) \\ &= -\sin(x)\sin(y) - d^2K^2\sin(x)\sin(y) \\ &= (-1 - d^2K^2)\sin(x)\sin(y) \end{aligned}$$

The correct value of K can now be found by setting the coefficients equal to each other giving that $d = -1 - d^2K^2$ or solving for K we get $K = \pm(\sqrt{-1-d})/|d|$. This tells us that the underlying functions take the form:

$$f(x) = \cos(x) \pm \sqrt{-1-d}\sin(x)$$

When $d = -1$ there is only one underlying function that will work, when $d < -1$ there are two distinct real underlying functions that will work and most interesting is that when $d > -1$ there will be two complex-valued underlying functions, even though the resultant function is real valued.

This situation is not too surprising given that K is the root of a quadratic, and in this example we have all possibilities for roots of a quadratic realized. (Note: complex valued functions can show up quite readily, for example, if $u(0,0) > 1/4$ then the underlying function must be complex valued if it exists.)

While the result does not apply when $d = 0$ it is interesting to note that when we put $d = 0$ in the final form the two resulting functions are $f(x) = \cos(x) \pm i\sin(x) = e^{\pm ix}$ which are exponential functions and are part of the homogeneous solutions.

3. A power form example

Given $u(x,y) = a(x+y)$ where $a \neq 0$ we will attempt to find the underlying function for $u(x,y)$ if it were a power form.

We will start by assuming that it is of the power form and then we can apply theorem 2 since $u(1,1) = 2a \neq 0$. This will give us the following result:

$$\begin{aligned} f(x) &= \frac{a(x+1) - a(x+1)}{2a} - Ka(x+1) \\ &= Ka(x+1) \end{aligned}$$

All that remains is to determine the correct value of K. We will proceed as before by comparing the original $u(x,y)$ with the one produced by the candidate underlying function.

$$\begin{aligned} a(x+y) &= u(x,y) \\ &= f(xy) - f(x)f(y) \\ &= Ka(xy+1) - [Ka(x+1)][Ka(y+1)] \\ &= -a^2K^2(x+y) + (aK - a^2K^2)(xy+1) \end{aligned}$$

This reduces down to two necessary equations that need to be satisfied in order for a solution to exist. Namely,

$$a = -a^2K^2, \qquad 0 = aK - a^2K^2,$$

which simplify to the following (recall $a \neq 0$)

$$-1 = aK^2, \qquad K = aK^2.$$

Combining the two equations together give that $K = -1$ and so it must be that $a = -1$. In particular, there is only one admissible value of a, and it has as an underlying function $f(x) = x + 1$.

This example shows some of the challenges of working with functions in these forms, it is possible that $u(x,y)$ might possess an underlying function, while a multiple of $u(x,y)$ might not.

4. An example with no underlying function

In the previous two examples we have demonstrated how to find an underlying function. We will now demonstrate the case where there is no underlying function. Specifically, we will demonstrate that their is no underlying function of an exponential form for $u(x,y) = d\cos(x)\cos(y)$ for any value of $d \neq 0$.

To do this we will start by assuming that it is of the exponential form and then by noting that $u(0,0) = d \neq 0$ we can use theorem 1 and get the following candidates, $f(x) = dK\cos(x)$ (by this time the method should be familiar enough that we can skip this intermediate step).

From this we get the following,

$$\begin{aligned} d\cos(x)\cos(y) &= u(x,y) \\ &= f(x+y) - f(x) - f(y) \\ &= dK\cos(x+y) - [dK\cos(x)][dK\cos(y)] \\ &= (dK - d^2K^2)\cos(x)\cos(y) - dK\sin(x)\sin(y) \end{aligned}$$

This gives us two equations that need to be simultaneously satisfied in order for an underlying function to exist, namely,

$$d = dK - d^2K^2, \qquad 0 = -dK.$$

Since $d \neq 0$ this implies by the second equation that $K = 0$ which in turn by the first equation implies $d = 0$, a contradiction and hence there exists no underlying function and so $u(x,y) = d\cos(x)\cos(y)$ is not of the Cauchy exponential form.

Conclusion

In this note we explored how to determine the underlying functions of the Cauchy exponential and Cauchy power forms. This in turn gave us a way to test as to whether or not a given $u(x,y)$ has an underlying function. Finally, several examples were given to demonstrate the application of this method, and some of the interesting features that the various forms have.

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