Computing inertia sets using atoms

Wayne Barrett[*] Steve Butler[†] H. Tracy Hall[‡] John Sinkovic[§]
Wasin So[¶] Colin Starr[‖] Amy Yielding[**]

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Abstract

We consider the problem of computing inertia sets for graphs. By using tools for combining the inertia sets of smaller graphs we can reduce this problem to understanding the inertia sets for three-connected graphs that are not joins. We term such graphs atoms and give the inertia sets for all atoms on at most seven vertices. This can be used to compute the inertia sets for all graphs on at most seven vertices.

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1 Introduction

In this paper we will be primarily dealing with simple graphs, i.e., graphs without loops or multiple edges, except where specified.

For a given graph G on the vertices $1, 2, \ldots, n$, let $\mathcal{S}(G)$ be the set of real symmetric matrices A whose off-diagonal entries a_{ij} are nonzero if and only if there is an edge joining i and j in the graph; the diagonal entries are allowed to be arbitrary. For a given matrix $A \in \mathcal{S}(G)$ the *partial inertia* of that matrix is the pair (p, q) where p is the number of positive

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[*] Department of Mathematics, Brigham Young University, Provo, UT 84602 (`wayne@math.byu.edu`)

[†] Department of Mathematics, UCLA, Los Angeles, CA 90095 (`butler@math.ucla.edu`).

[‡] Department of Mathematics, Brigham Young University, Provo, UT 84602 (`H.Tracy@gmail.com`)

[§] Department of Mathematics, Brigham Young University, Provo, UT 84602 (`johnsinkovic@gmail.com`)

[¶] Department of Mathematics, San Jose State University, San Jose, CA 95192 (`so@math.sjsu.edu`)

[‖] Mathematics Department, Willamette University, Salem, OR 97301 (`cstarr@willamette.edu`)

[**] Mathematics Program, College of Arts and Sciences, Eastern Oregon University, La Grande, OR 97850 (`ayielding@eou.edu`)

eigenvalues and q is the number of negative eigenvalues. The *inertia set* for G, denoted $\mathcal{I}(G)$, is the set of all partial inertias for the matrices in $\mathcal{S}(G)$, which can be viewed as a subset of the plane (see [2, 4]).

The problem of determining the inertia set for a given graph, known as the inverse inertia problem, grew as a natural generalization of the problem of determining the minimum rank of a graph. For example, $\min_{(a,b)\in\mathcal{I}(G)}(a+b)$ is the minimum rank of the graph. So given the inertia set, the minimum rank is easily computed. The converse is not true, since it is possible for the inertia set of a graph to be missing several entries lying on or above the minimum rank line.

There are several reductions for computing inertia sets of graphs in terms of smaller graphs. In particular, if a graph is disconnected, has a cut-vertex, has a 2-separation or is a join, then the inertia set for the graph can be determined using the inertia sets of smaller graphs. (The tools for how to do this are gathered in Section 2; we also remark that these are not the only reductions possible, but all the reductions we have selected can be applied unconditionally). Armed with these tools we can reduce the problem of determining inertia sets to graphs that are 3-connected but are not joins. We dub these graphs *atoms*, since they cannot be broken down further.

In this note we compute the inertia sets for all atoms on at most seven vertices (this is done in Section 3). Combined with the tools of Section 2 one can then compute the inertia sets for all graphs on seven vertices, as well as for many graphs on eight or more vertices. We also give some open problems in Section 4.

Throughout this paper, we will refer to graphs using their listing in the Atlas [14].

Some characteristics of inertia sets

Since $A \in \mathcal{S}(G)$ if and only if $-A \in \mathcal{S}(G)$, we have $(a,b) \in \mathcal{I}(G)$ if and only if $(b,a) \in \mathcal{I}(G)$. In other words the inertia sets also have symmetry across the line $y = x$. Further, inertia sets do not have holes in the sense of the following "Northeast Lemma," which says that if a point is in the inertia set then everything above and to the right (up to the dimension constraint) is also in the set.

Proposition 1 (Barrett, Hall, Loewy [2])**.** *If $(a,b) \in \mathcal{I}(G)$ then $(a+s, b+t) \in \mathcal{I}(G)$ where s and t are nonnegative integers and $(a+s)+(b+t) \le n$.*

One consequence of this is that inertia sets tend to contain (and sometimes are) large trapezoidal regions. We denote trapezoidal regions by

$$T[k,\ell] = \{(a,b) : k \le a+b \le \ell, a \ge 0, b \ge 0\}.$$

For example, our above comment about minimum rank says that $\mathcal{I}(G) \subseteq T[\mathrm{mr}(G), n]$, where $\mathrm{mr}(G)$ is the minimum rank of the graph.

There are several ways to combine inertia sets together. The simplest is to take a sum, i.e.,

$$\mathcal{I}(H) + \mathcal{I}(G) = \big\{(a+c, b+d) : (a,b) \in \mathcal{I}(H) \text{ and } (c,d) \in \mathcal{I}(G)\big\}.$$

In this case we might end up with terms that are unfeasible, i.e., points (a, b) with $a+b > n$ where n is the number of vertices in the graph. In this case we simply will remove such points from the sum set S; this is denoted by $[S]_n$.

2 Tools for computing inertia sets from proper subgraphs

In this section we gather together several tools that allow us, in some special cases, to compute inertia sets for graphs in terms of inertia sets of smaller graphs. The first set of tools tell us that if a graph is sufficiently sparse, i.e., has connectivity at most 2, then we can decompose the computation of the inertia sets into smaller parts.

This is easiest if the connectivity is 0 or 1.

Theorem 2 (Barrett, Hall, Loewy [2])**.** *1. Let the graph G be a disjoint union of two graphs G_1 and G_2. Then $\mathcal{I}(G) = \mathcal{I}(G_1) + \mathcal{I}(G_2)$.*

2. Let the graph $G = G_1 \oplus_v G_2$, where G has n vertices; i.e., let G_1 and G_2 be disjoint graphs on at least two vertices, each with a vertex labeled v, and let G be the graph defined by identifying the vertex labeled v in G_1 and G_2 together. Then

$$\mathcal{I}\big(G_1 \underset{v}{\oplus} G_2\big) = [\mathcal{I}(G_1) + \mathcal{I}(G_2)]_n \cup [\mathcal{I}(G_1 - v) + \mathcal{I}(G_2 - v) + \{(1,1)\}]_n.$$

We now turn to graphs with connectivity equal to 2. It is necessary to use graphs with multiple edges in order to state this result so we need to first expand our definition of the inertia set to such graphs.

For a graph G with multiple edges, we say that the simple graph H is a simple realization of G, denoted $H \prec G$, if H can be obtained from G by replacing each multiedge of G by either one edge or no edge. So if G has k multiedges, there are 2^k simple realizations of G. Then we define

$$\mathcal{I}(G) = \bigcup_{H \prec G} \mathcal{I}(H).$$

Note that if G is a simple graph, this reduces to the definition we already have.

Example 3. $\mathcal{I}\big($$\big) = \mathcal{I}(K_3) \cup \mathcal{I}(P_3) = T[1,3] \cup T[2,3] = T[1,3]$.

We also need the idea of vertex identification used in [11]. Given a graph G and vertices v_1, v_2 of G, we construct a new graph G/v_1v_2 by removing the edges $\{v_1, v_2\}$, if present in G, and then identifying v_1, v_2 to one vertex v. In G/v_1v_2, each edge $\{u, v_i\}$, for $i = 1$ and 2, becomes an edge $\{u, v\}$ in G/v_1v_2. When both edges are present this will create a multiedge between u and v in G/v_1v_2. For example, identifying two vertices in K_n results in $n-2$ multiedges.

(a) G (b) G/v_1v_2

Figure 1: A graph and the resulting graph from vertex identification

Example 4. *Consider the graph shown in Figure 1.*

According to our definition of the inertia of a multiset,

$\mathcal{I}(G/v_1v_2) = \mathcal{I}($  $) \cup \mathcal{I}($  $)$

$$= T[3,5] \cup \big(\{(1,1),(2,1),(1,2)\} \cup T[4,5]\big)$$

$$= \{(1,1)\} \cup T[3,5].$$

We note that in this example $\mathcal{I}(G/v_1v_2)$ is larger than the inertia set of either graph in the union.

We are now ready to state the result for graphs with connectivity 2.

Theorem 5 (van der Holst [10])**.** *Let (G_1, G_2) be a 2-separation of a graph G with n vertices and let $\{v_1, v_2\} = V(G_1) \cap V(G_2)$. Let H_1 and H_2 be obtained from G_1 and G_2, respectively, by adding an edge (possibly creating a multiedge) between v_1 and v_2. Then*

$$\begin{aligned}\mathcal{I}(G) = \; & [\mathcal{I}(G_1) + \mathcal{I}(G_2)]_n \cup [\mathcal{I}(G_1/v_1v_2) + \mathcal{I}(G_2/v_1v_2) + \{(1,1)\}]_n \cup \\ & [\mathcal{I}(G_1 - v_1) + \mathcal{I}(G_2 - v_1) + \{(1,1)\}]_n \cup \\ & [\mathcal{I}(G_1 - v_2) + \mathcal{I}(G_2 - v_2) + \{(1,1)\}]_n \cup \\ & [\mathcal{I}(G_1 - \{v_1, v_2\}) + \mathcal{I}(G_2 - \{v_1, v_2\}) + \{(2,2)\}]_n \cup \\ & [\mathcal{I}(H_1) + \mathcal{I}(H_2)]_n.\end{aligned}$$

This decomposition tool for the 2-separation is similar to the rule used to compute the minimum rank for graphs with a 2-separation (see [11, Corollary 15]). We note that the inertia sets for many of these sparse graphs can be calculated much more quickly using results in [4], but our aim has been to note that such graphs are all covered by known theorems.

The other tool is for graphs that are sufficiently dense; i.e., graphs that have a dominating vertex or can be written as a join of two graphs. Then we can again decompose the computing of the inertia set into smaller parts.

Theorem 6 (Barrett, Hall, van der Holst [1])**.** *Let G be a graph with t isolated vertices. Then*

$$\mathcal{I}^{\nearrow}(G \vee K_1) = \mathcal{I}^{\nearrow}(G) + \mathcal{I}^{\nearrow}(K_{1,t}).$$

Let G_1 and G_2 be graphs and $G_1 \vee G_2$ their join. Then

$$\mathcal{I}^{\nearrow}(G_1 \vee G_2) \setminus \{(1,1)\} = [\mathcal{I}^{\nearrow}(G_1 \vee K_1) \cap \mathcal{I}^{\nearrow}(G_2 \vee K_1)] \setminus \{(1,1)\}.$$

Note that in the last case the rule will not determine if $(1,1)$ should be in our inertia set. However, a graph has inertia $(1,1)$ if and only if it has minimum rank at most two (see [4, Corollary 3.12]). These graphs have been completely characterized and are easy to identify by examining their complement (see [3]).

3 Inertia sets for atoms with at most seven vertices

In this section we determine the inertia sets for all atoms on at most seven vertices. Together with the decomposition rules from Section 2, this will allow us to determine the inertia sets for all graphs up to seven vertices. There are 2 atoms on six vertices and 58 atoms on seven vertices. Atoms are connected and have connected complement (i.e., none is a join).

Proposition 7 ([13, page 10])**.** *If a graph on at least two vertices and its complement are connected then the graph contains P_4 as an induced subgraph.*

In particular, since P_4 has minimum rank three then an atom on more than one vertex must have minimum rank at least three.

Atoms on n vertices with K_{n-2} minors

Two useful tools that will help us compute the inertia sets for our atoms are the parameters μ and ν introduced by Colin de Verdière. We will not define them here, but summarize the important properties of these parameters below.

Proposition 8 ([5, 6, 7, 9])**.** *The parameters μ and ν satisfy the following conditions:*

(a) If H is a minor of G, then $\mu(H) \le \mu(G)$ and $\nu(H) \le \nu(G)$.

(b) $\mu(K_n) = \nu(K_n) = n - 1$.

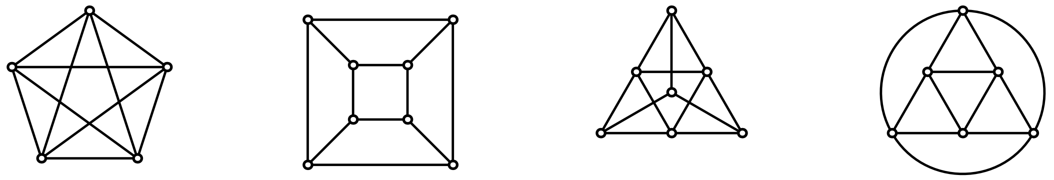
(c) If $\mu(G) \ge k$ then $(n-k-1, 1)$ is in $\mathcal{I}(G)$. If $\nu(G) \ge k$ then $(n-k, 0)$ is in $\mathcal{I}(G)$.

(d) $\mu(G) \ge 4$ if and only if the graph G is not planar. $\nu(G) = 4$ for K_5, Q_3, $Q_3 Y\Delta$ and $K_{2,2,2}$ (see Figure 2).

The graph $Q_3 Y\Delta$ is obtained by starting with Q_3 and removing one of the vertices of degree 3 (the "Y") and placing in a triangle (the "Δ") on the set of vertices the Y was previously adjacent to.

Using Proposition 8 we can now give one general family of graphs for which we can compute $\mathcal{I}(G)$.

Proposition 9. *If an atom G on n vertices contains K_{n-2} as a minor, then $\mathcal{I}(G) = T[3, n]$.*

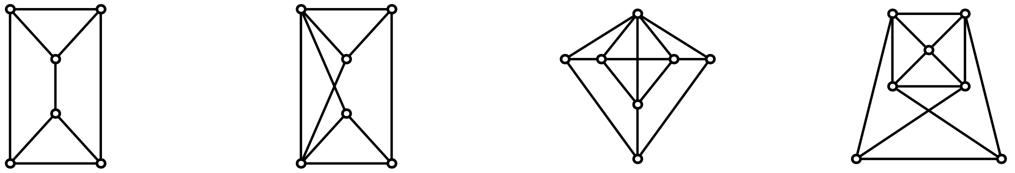


(a) K_5 = `G52` (b) Q_3 = `C8` (c) $Q_3Y\Delta$=`G1005` (d) $K_{2,2,2}$ = `G204`

Figure 2: Graphs with $\nu(G) = 4$

Proof. Since, as noted above, the minimum rank is at least 3, we have $\mathcal{I}(G) \subseteq T[3, n]$. To show we have equality it suffices to show that $(2, 1)$ and $(3, 0)$ are in our inertia set and then apply symmetry and Proposition 1. Since K_{n-2} is a (connected) minor of our (connected) atom G, by Proposition 8 we have $\mu(G) \geq \mu(K_{n-2}) = n - 3$ so that $(2, 1)$ is in our inertia set. Again by Proposition 8 we have $\nu(G) \geq \nu(K_{n-2}) = n - 3$ so that $(3, 0)$ is in our inertia set. □

Proposition 9 handles both of the atoms on six vertices, `G174` and `G188`, as well as 28 atoms on seven vertices, `G1000`, `G1001`, `G1090`, `G1094`, `G1096`, `G1098`, `G1100`, `G1102`, `G1103`, `G1105`, `G1147`, `G1153`, `G1155`, `G1156`, `G1157`, `G1158`, `G1159`, `G1161`, `G1162`, `G1165`, `G1166`, `G1168`, `G1170`, `G1194`, `G1196`, `G1200`, `G1201` and `G1209`. See Figures 3 and 8 for examples of some of these graphs.



(a) `G174` (b) `G188` (c) `G1096` (d) `G1105`

Figure 3: Examples of atoms for which Proposition 9 applies

Nonplanar atoms on seven vertices with induced $K_2 \cup 2K_1$

For graphs on seven vertices, Proposition 9 shows that if a graph has K_5 as a minor then the inertia set is the trapezoid $T[3, 7]$. The fact that K_5 is a minor is used in two ways: to show $\mu \geq 4$ and to show $\nu \geq 4$. However, this does not cover all nonplanar atoms on seven vertices since a graph can also be nonplanar because $K_{3,3}$ is a minor. While this will not affect μ, it might affect ν. The following proposition will handle the remaining nonplanar atoms on seven vertices.

Proposition 10. *If an atom G on seven vertices is nonplanar and has $K_2 \cup (2K_1)$ as an induced subgraph, where the vertices of the K_2 are not twins (i.e., they have a different set of neighbors), then $\mathcal{I}(G) = T[4,7] \cup \big\{(2,1),(1,2)\big\}$.*

Proof. As in Proposition 9 we have that the minimum rank of the graph is at least 3 so that $\mathcal{I}(G) \subseteq T[3,7]$. To establish the result we need to show that $(2,1)$ and $(4,0)$ are in the inertia set while $(3,0)$ is not. Since the atom is nonplanar we have by Proposition 8 that $\mu(G) \geq 4$, showing $(2,1)$ is in our inertia set. Since the atom is nonplanar it will have K_4 as a minor (since K_4 is a minor both of K_5 and $K_{3,3}$) and so we have $\nu(G) \geq \nu(K_4) = 3$, showing $(4,0)$ is in our inertia set.

To show that $(3,0)$ is not in our inertia set, suppose that by contradiction it were. Then there would be a way to associate 3-dimensional nonzero vectors with our vertices where non-adjacent vertices were orthogonal. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the two vectors associated with the K_1 vertices and $\mathbf{y}_1$ and $\mathbf{y}_2$ be the two vectors associated with the K_2 vertices of the induced $K_2 \cup (2K_1)$. Since $\mathbf{y}_1$ and $\mathbf{y}_2$ would both have to be orthogonal to $\mathbf{x}_1$ and $\mathbf{x_2}$, we would be forced to conclude that $\mathbf{y}_1$ and $\mathbf{y}_2$ are parallel. But this would force them to have the same set of neighbors (i.e., they are orthogonal or not orthogonal to the same set of vectors), a contradiction. $\square$

Proposition 10 handles the following 7 atoms on seven vertices, `G876`, `G994`, `G1003`, `G1004`, `G1084`, `G1085` and `G1092`. See Figure 4 for examples of some of these graphs.

There are other ways to show that the point $(3,0)$ is not in the inertia set for these graphs instead of using the induced $K_2 \cup (2K_1)$. For instance, each of these graphs contain the graph on six vertices `G149` as an induced subgraph. Since it is known that the minimum semidefinite rank of `G149` is 4 (see [4]), the needed result would again follow.

Both approaches are similar in that we are finding some induced subgraph with large minimum semidefinite rank to help give a lower bound for the semidefinite rank of the graph. This idea used in Propositon 10 can be generalized as follows: if G is a connected graph and has an induced $(sK_2) \cup (tK_1)$, where none of the vertices in the copies of K_2 are twins, then the minimum semidefinite rank is at least $2s + t$.

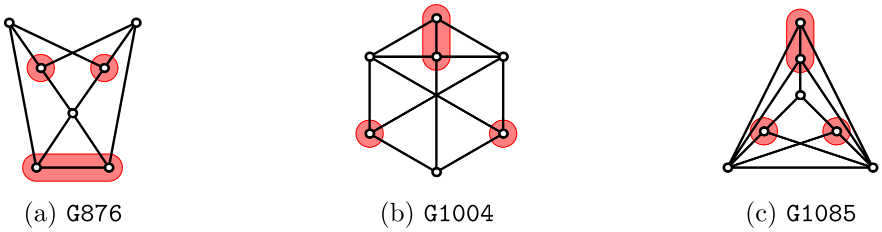

(a) `G876` (b) `G1004` (c) `G1085`

Figure 4: Examples of atoms for which Proposition 10 applies (the $K_2 \cup (2K_1)$ is marked)

Planar atoms on seven vertices with $\nu(G) \leq 3$

We now turn our attention to the planar atoms on seven vertices. We start with the following two results.

Theorem 11 (van der Holst [8]). *For a 3-connected graph G, the following are equivalent:*

(i) G is a partial 3-path

(ii) G has no K_5, $K_{3,3}$, $K_{2,2,2}$, Q_3 or $Q_3Y\Delta$ minor

Theorem 12 (Kempton [12, Theorem 4.23]). *If G is a k-connected partial k-path on n vertices then $\mathcal{I}(G) = T[n-k, n]$.*

Corollary 13. *If G is a planar atom on seven vertices with no $K_{2,2,2}$ or $Q_3Y\Delta$ minor then $\mathcal{I}(G) = T[4, 7]$*

Proof. First we note that since G is an atom it is 3-connected. Also, since G is planar it does not have a K_5 or $K_{3,3}$ minor. By assumption it has no $K_{2,2,2}$ or $Q_3Y\Delta$ minor, and since it only has seven vertices it cannot have a Q_3 minor. By Theorem 11 the atom is a 3-connected partial 3-path, so that by Theorem 12 we can conclude that $\mathcal{I}(G) = T[4, 7]$. ☐

In practice, to check that there is no $K_{2,2,2}$ minor one can try to show that no minor will have six vertices of degree at least 4. To show that there is no $Q_3Y\Delta$ minor (i.e., $Q_3Y\Delta$ is not a subgraph) it is usually simplest to look at the complement. Since the complement of $Q_3Y\Delta$ has no four cycle then any graph on seven vertices whose complement contains a four cycle must not contain $Q_3Y\Delta$ as a minor.

Corollary 13 handles the following 17 atoms on seven vertices: `G875`, `G877`, `G992`, `G993`, `G997`, `G998`, `G999`, `G1006`, `G1082`, `G1083`, `G1089`, `G1091`, `G1093`, `G1097`, `G1101`, `G1145` and `G1154`. See Figure 5 for examples of some of these graphs.

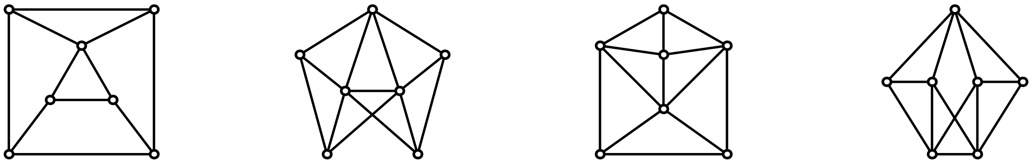
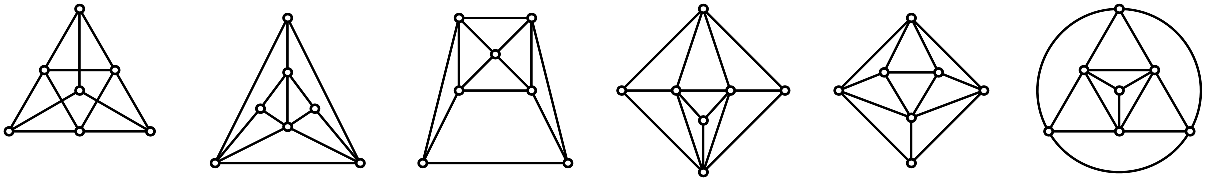

(a) `G877` (b) `G1083` (c) `G1097` (d) `G1101`

Figure 5: Examples of atoms for which Corollary 13 applies



(a) `G1005` (b) `G1095` (c) `G1104` (d) `G1146` (e) `G1167` (f) `G1205`

Figure 6: Planar atoms on seven vertices with $\nu = 4$

Planar atoms on seven vertices with $\nu(G) = 4$

There remain 6 atoms on seven vertices, all of which are planar: `G1005`, `G1095`, `G1104`, `G1146`, `G1167` and `G1205`. All of these atoms contain $Q_3 Y\Delta$ as a subgraph; in addition, the atoms `G1104`, `G1167` and `G1205` contain $K_{2,2,2}$ as a minor. These graphs are shown in Figure 6 as represented in the Atlas [14]. We now show that all of these graphs have $\mathcal{I}(G) = T[3, 7]$.

As before, we must have that the inertia is contained in $T[3, 7]$ (i.e., the minimum rank is at least 3). To show equality we must show that $(3, 0)$ and $(2, 1)$ are in the inertia sets. But since $\nu(Q_3 Y\Delta) = \nu(K_{2,2,2}) = 4$ we can conclude that $\nu(G) \geq 4$ for all of these graphs, i.e., they all contain $(3, 0)$ in their inertia set. To show that they have $(2, 1)$ in their inertia sets we give explicit matrices with this inertia. We have the following set of matrices which clearly have inertia $(2, 1)$ (because of the sign of the entries in the diagonal matrix in the middle term), and a computation shows that each matrix is an admissible matrix for the given atom (in Figure 7 we have drawn these atoms to both emphasize how $Q_3 Y\Delta$ is a subgraph and to label the vertices as they correspond to the matrices generated).

$$\text{For } \texttt{G1005}\text{:}\quad A = B_{\texttt{G1005}}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} B_{\texttt{G1005}} \quad \text{where} \quad B_{\texttt{G1005}} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 \end{pmatrix}$$

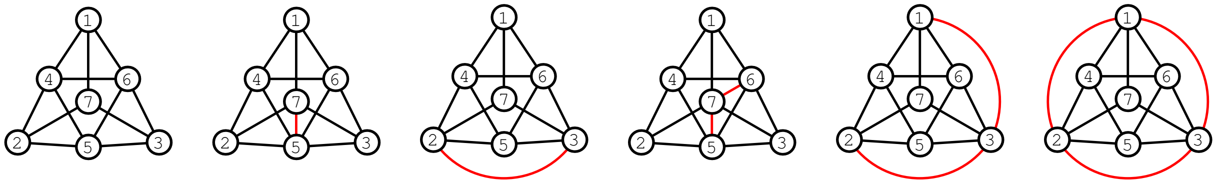
$$\text{For } \texttt{G1095}\text{:}\quad A = B_{\texttt{G1095}}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} B_{\texttt{G1095}} \quad \text{where} \quad B_{\texttt{G1095}} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & -1 \end{pmatrix}$$

$$\text{For } \texttt{G1104}\text{:}\quad A = B_{\texttt{G1104}}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} B_{\texttt{G1104}} \quad \text{where} \quad B_{\texttt{G1104}} = \begin{pmatrix} 2 & 1 & 0 & 1 & 1 & 1 & 0 \\ -1 & -1 & 1 & 0 & 2 & 1 & 0 \\ 1 & 3 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{For } \texttt{G1146}\text{:}\quad A = B_{\texttt{G1146}}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} B_{\texttt{G1146}} \quad \text{where} \quad B_{\texttt{G1146}} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 \end{pmatrix}$$

For `G1167`: $$A = B_{\texttt{G1167}}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} B_{\texttt{G1167}} \quad \text{where} \quad B_{\texttt{G1167}} = \begin{pmatrix} 2 & 1 & 0 & 1 & 1 & 1 & 0 \\ -1 & -1 & 1 & 0 & 2 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

For `G1205`: $$A = B_{\texttt{G1205}}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} B_{\texttt{G1205}} \quad \text{where} \quad B_{\texttt{G1205}} = \begin{pmatrix} 2 & 1 & 0 & 1 & 1 & 1 & 0 \\ -1 & -1 & 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$



(a) `G1005` (b) `G1095` (c) `G1104` (d) `G1146` (e) `G1167` (f) `G1205`

Figure 7: Alternative drawings for planar atoms on seven vertices with $\nu = 4$

An alternative method to compute the inertia sets for two of the graphs (`G1146` and `G1205`) is by using Theorem 8.1 and Table 9.1 in [4].

4 Concluding remarks

In this note we have been able to combine several tools that allow us to compute the inertia sets for graphs on seven or fewer vertices. The two main parts of this technique were rules to break the problem into smaller components and then to identify the inertia sets on these smaller components. One could continue this, and consider the inertia sets for atoms with eight or more vertices. This would require some new insights since not only are there far more atoms on eight vertices (1779 atoms), but also the tools we are using are near the edge of what is known. For example, it is not known which graphs are forbidden minors to have $\nu \leq 4$, which would be helpful in understanding the inertia sets for atoms.

A different approach might be to find a formula for computing the inertia for graphs that have a 3-separator (or in general a k-separator); i.e., find a way to split the atoms. If such a formula were known then the number of atoms would drop to 2 atoms on seven vertices (`G1170` and `G1209`, see Figure 8), and 176 atoms on eight vertices (many of which could be handled by Proposition 9), and so on. But historically, splitting the atom has proven to be difficult.

We also note that all the graphs on seven vertices are inertially balanced, which is to say that there is a matrix that achieves the minimum rank and for which the difference between the number of positive and negative eigenvalues is at most 1. It is known that there is a matrix on 12 vertices that is not inertially balanced (see [2]); and one would expect that the smallest not inertially balanced graph is an atom, suggesting that atoms might make for some interesting examples for inertia sets.

(a) `G1170`

(b) `G1209`

Figure 8: Atoms on seven vertices that are 4-connected

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