

Relating the arboricity with the chromatic number of a graph

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In this note we will assume that we are working with a simple undirected graph G without loops or multiple edges. The arboricity of G , denoted $a(G)$, is the minimum number of forests that the edges of G can be edge-decomposed into (i.e., there is some collection of $a(G)$ subgraphs of G where each subgraph is a forest and each edge in G is in exactly one such subgraph).

On the other hand the chromatic number of G , denoted $\chi(G)$, is the minimum number of colors needed to color the vertices so that no two adjacent vertices are the same color. Equivalently, the chromatic number is the minimum number of independent sets that the vertices can be partitioned into.

Theorem. $\chi(G) \leq 2a(G)$.

This is a consequence of the following lemma.

Lemma. Let G be a graph on n vertices. Then the vertices can be labeled v_1, v_2, \dots, v_n such that for any vertex v_i it is adjacent to at most $2a(G) - 1$ vertices v_j where $j < i$.

Proof of theorem. Using the lemma we arrange the vertices v_1, \dots, v_n so that v_i is adjacent to at most $2a(G) - 1$ vertices v_j where $j < i$. We now color the vertices in order with $2a(G)$ colors by letting v_i be colored with any color that is not used in the vertices v_j for which $j < i$ and v_j is adjacent to v_i . It is simple to verify that this produces a valid $2a(G)$ coloring of the vertices. \square

Proof of lemma. We proceed by induction on n . The result trivially holds for $n = 1$. Now suppose it holds for $n - 1$. Since the degree sum of a forest on n vertices is at most $2n - 2$ we have,

$$\frac{\sum_{v \in G} d_v}{n} \leq \frac{k(2n - 2)}{n} < 2k,$$

where d_v is the degree of vertex v . This implies that the average degree of G is at most $2k - 1$ and so in particular there is some vertex with degree at most $2k - 1$. Label this vertex v_n , remove it from the graph, the remaining graph can also be decomposed into k forests and so by induction we can label the remaining vertices. It is now easy to check that the conditions of the lemma are satisfied. \square

Comments. It is known that for a planar graph G , $a(G) \leq 3$ and so by the above theorem every planar graph is 6-colorable, of course this can be improved. Next we note that there is no bound for $a(G)$ in terms of $\chi(G)$, for example $K_{n,n}$ needs at least $n^2/(2n - 1)$ forests to decompose its edges but has chromatic number 2. We also note that the bound given in the theorem above is tight since K_{2n} can be edge decomposed into n paths (an example is shown below for $n = 4$) and has chromatic number $2n$.

