Anti-covers of double edge coverings of graphs

Steve Butler

May 5, 2007

1 Introduction

Spectral graph theory looks at the connections between the structure of a graph and the eigenvalues of various matrices associated with the graph. By examining these eigenvalues (or the spectrum) various properties of a graph can be determined (or at least bounded). When two graphs have several eigenvalues in common it can often be traced to some shared structure. One common example of this is when two graphs cover the same graph or share the same equitable partition.

Consider the graphs in Figure 1, these three graphs share $\frac{1}{6}(5-\sqrt{7})$, $\frac{1}{6}(7-\sqrt{7})$, $\frac{1}{6}(5+\sqrt{7})$ and $\frac{1}{6}(7+\sqrt{7})$ as (nontrivial) eigenvalues of the normalized Laplacian. It will turn out that this is not a result of a shared covering or equitable partition but rather a shared *anti-covering* of various coverings of these graphs.

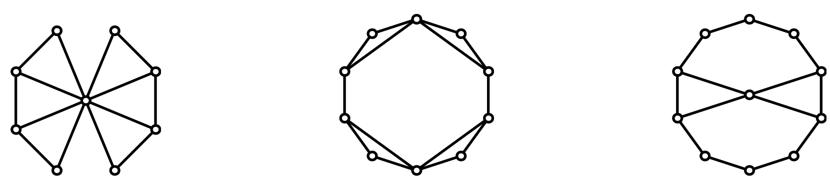

Figure 1: Three graphs sharing some common eigenvalues.

We proceed as follows, in the remainder of the introduction we introduce the types of graphs and matrices we will be considering. In Section 2 we define double edge coverings and establish a basic property. In Section 3 we consider restricted double edge coverings and the adjacency matrix. In this section we will first introduce anti-coverings and use them to establish the following result (among others).

Theorem 1. *Let A be a real $n\times n$ symmetric matrix and let $|A|$ be the $n\times n$ matrix defined entrywise by $|A|_{i,j} = |A_{i,j}|$ (i.e., the absolute value of the corresponding entry of A). Then there exists a nonnegative $2n\times 2n$ symmetric matrix B so that the eigenvalues of B are the union of the eigenvalues of A and $|A|$ (counting multiplicity).*

In Section 4 we consider double edge coverings and the normalized Laplacian. We will see that the eigenvalues of the normalized Laplacian are more sensitive in detecting common anti-coverings of graphs. Finally in Section 5 we give some concluding comments.

1.1 Matrices associated with graphs

Except where stated otherwise, we will be considering graphs without isolated vertices and with a nonnegative weight function, w, on the edges (which possibly includes loops). For a graph G with vertex set $V = V(G)$ and edge set $E = E(G)$, the weight function $w : V \times V \rightarrow [0, \infty)$ is such that $w(u, v) = w(v, u)$ (i.e., the graph is undirected) and by convention $u \sim v$ (u is adjacent to v) if and only if $w(u, v) > 0$. The degree of a vertex is the sum of the weights of the edges incident to that vertex, i.e., $d(u) = \sum_{v \sim u} w(v, u)$. (Note that a simple graph corresponds to the case when $w(u, v) \in \{0, 1\}$.)

We will be considering two matrices associated with weighted graphs. The first is the adjacency matrix which is defined entrywise by $A_{u,v} = w(u, v)$. The second is the normalized Laplacian defined by $\mathcal{L} = D^{-1/2}(D - A)D^{-1/2}$ where D is the diagonal degree matrix, i.e., $D_{u,u} = d(u)$.

Since we will be examining the eigenvalues of these matrices it is also important to consider how eigenvectors and eigenvalues relate to the graph. For the adjacency matrix A, if $\mathbf{x}$ is an eigenvector then the relationship $A\mathbf{x} = \lambda\mathbf{x}$ translates into the condition that at each vertex v

$$\sum_{u \sim v} w(u, v)\mathbf{x}(u) = \lambda\mathbf{x}(v). \tag{1}$$

For the normalized Laplacian instead of focusing on the eigenvectors we will focus on the *harmonic* eigenvectors. Namely if $\mathbf{y}$ is an eigenvector of $\mathcal{L}$ associated with λ then the harmonic eigenvector is $\mathbf{x} = D^{-1/2}\mathbf{y}$. The relationship $\mathcal{L}\mathbf{y} = \lambda\mathbf{y}$ then becomes (after some simplifying) $(D - A)\mathbf{x} = \lambda D\mathbf{x}$ which translates into the condition that at each vertex v

$$d(v)\mathbf{x}(v) - \sum_{u \sim v} w(u, v)\mathbf{x}(u) = \lambda d(v)\mathbf{x}(v). \tag{2}$$

We will be showing that two graphs share common eigenvalues by showing that relationship (1) or (2) (depending on which matrix we are considering) holds for a given value of λ on both graphs.

2 Double edge coverings

Bya covering we mean a map from a graph G to a smaller graph $\hat{G}$ (the "covered" graph) such that the (local) structure of the larger graph is also present in the smaller graph. Since these graphs then share similar local structure they also share some common eigenvalues. This can be used to compute (a portion of) the spectrum of a large graph by computing the spectrum of a smaller graph.

We will be focusing on a special type of covering where each edge in the smaller graph gets covered twice by an edge in the larger graph. More precisely, given two weighted graphs G and $\hat{G}$ we say that G is a *double edge covering* of $\hat{G}$ if there exists an onto map $\pi : V(G) \rightarrow V(\hat{G})$ satisfying the following conditions:

(i) if $u \sim v$ in G then $\pi(u) \sim \pi(v)$ in $\hat{G}$ and further $w\big(\pi(u), \pi(v)\big) = w(u, v)$;

(ii) if $\pi(u) \sim \hat{w}$ in $\hat{G}$ then there is some vertex v in G so that $u \sim v$ and that $\pi(v) = \hat{w}$;

(iii) for each (ordered) edge $(\hat{w}, \hat{z})$ in $\hat{G}$ there are exactly two (ordered) edges (p, q), (r, s) in G so that $\big(\pi(p), \pi(q)\big) = \big(\pi(r), \pi(s)\big) = (\hat{w}, \hat{z})$.

Condition (i) insures that the map π preserves adjacency (with the correct weight), condition (ii) insures that we can lift edges incident to a vertex in $\hat{G}$ up to edges incident to a preimage of the vertex (this shows that locally $\hat{G}$ is preserving some structure of G). Condition (iii) requires that each edge is covered twice. Our reason for having the edges ordered in (iii) is to deal with the creation of loops. Namely, if $u \neq v$, $u \sim v$ and $\pi(u) = \pi(v)$ then we would have a loop at $\pi(u)$ in $\hat{G}$, by our convention the loop is double covered by (u, v) and (v, u). We will give several examples of double edge coverings in Sections 3 and 4.

It should be noted that not every covering of G to $\hat{G}$ which covers each edge exactly twice will qualify as a double edge covering. For instance, consider the double edge covering indicated in Figure 2 where $a_i \rightarrow a$, $b_i \rightarrow b$ and $c_i \rightarrow c$. It is simple to verify that this covering does not satisfy property (ii). A computation will show that these two graphs also share no common eigenvalues for either the adjacency or normalized Laplacian matrices (which as we will see below would occur for a double edge covering).

Since the graph is onto, every vertex in $\hat{G}$ is covered by at least one vertex in G, while properties (ii) and (iii) combined show that each vertex is covered at most twice. This gives two possibilities for the size of the preimage of a vertex, as the following lemma shows the two different possibilities have quite different behavior.

Lemma 2. *Let G be a double edge covering of $\hat{G}$ under the map π, and $\pi(v) = \hat{v}$.*

- *If $\big|\pi^{-1}(\hat{v})\big| = 2$ and $u \sim v \sim w$ with $u \neq w$ in G, then $\pi(u) \neq \pi(w)$.*

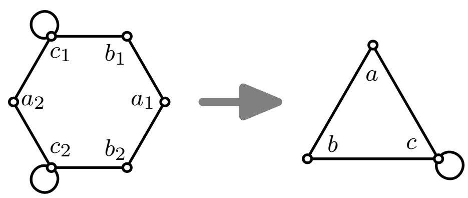

Figure 2: A non-example of an edge double covering.

- *If* $\left|\pi^{-1}(\hat{v})\right| = 1$ *and* $u \sim v$*, then there is some* w *such that* $v \sim w$*,* $u \neq w$ *and* $\pi(u) = \pi(w)$*.*

So when a vertex has two preimages there is a 1-to-1 correspondence between edges incident to v and $\hat{v}$. While if a vertex has only a single preimage the edges incident to v in G map 2-to-1 to the edges incident to $\hat{v}$ in $\hat{G}$, intuitively in this case the edges incident to v "fold" over.

Proof. First consider the case $\pi^{-1}(\hat{v}) = \{v, v'\}$. Suppose that $u \sim v \sim w$, then by property (i) $\pi(u) \sim \hat{v}$ so by property (ii) there exists some vertex z in G so that (z, v') and (u, v) are distinct edges in G both covering the same edge in $\hat{G}$. Now if $\pi(u) = \pi(w)$ then (w, v) would be a third edge in G which also covers but this contradicts (iii). Therefore we have that $\pi(u) \neq \pi(w)$.

Now consider the case $\pi^{-1}(\hat{v}) = \{v\}$. If $u \sim v$ then by property (i) $\pi(u) \sim \hat{v}$. By property (iii) this edge is double covered and since $\hat{v}$ has only one preimage the two edges in G which double cover it are (u, v) and (w, v) (for some w). But now note that $v \sim w$, $u \neq w$ and that $\pi(u) = \pi(w)$ as needed. $\square$

We can use the map π to lift vectors from $\hat{G}$ to G, namely if $\hat{\mathbf{x}}$ is a vector on the graph $\hat{G}$ we can let $\mathbf{x}$ be a vector on G defined by $\mathbf{x}(v) = \hat{\mathbf{x}}\big(\pi(v)\big)$. Similarly given a vector $\mathbf{x}$ on G we can push it down to a vector $\hat{\mathbf{x}}$ on $\hat{G}$ by letting

$$\hat{\mathbf{x}}(\hat{v}) = \frac{1}{|\pi^{-1}(\hat{v})|} \sum_{w \in \pi^{-1}(\hat{v})} \mathbf{x}(w).$$

From the definition it is easy to see that if we first lift $\hat{\mathbf{x}}$ up to G and then push it down to $\hat{G}$ we will get back the vector we started with, on the other hand if we first push down a vector $\mathbf{x}$ and then lift it back up we will not always have what we started with.

3 Double coverings and the adjacency matrix

In this section we will assume that in addition to being a double edge covering that each vertex has two preimages. We will call such coverings double coverings. Two examples of the six cycle double covering a graph are shown in Figure 3.

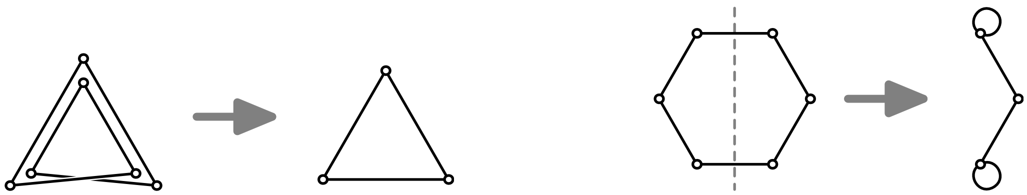

Figure 3: Two examples of the six cycle double covering a graph.

Lemma 3. *Suppose that G is a double cover of $\hat{G}$. Then the eigenvalues of the adjacency matrix of $\hat{G}$ (counting multiplicity) are also eigenvalues of the adjacency matrix of G.*

Proof. Let $\hat{\mathbf{x}}$ be a nonzero eigenvector of $\hat{G}$ associated with eigenvalue λ and let $\mathbf{x}$ be the lifted vector on G described at the end of Section 2. Then for any $v \in G$,

$$\sum_{u\sim v} w\big(u,v\big)\mathbf{x}(u) = \sum_{\hat{u}\sim\pi(v)} w\big(\hat{u},\pi(v)\big)\hat{\mathbf{x}}(\hat{u}) = \lambda\hat{\mathbf{x}}\big(\pi(v)\big) = \lambda\mathbf{x}(v),$$

Since the lifted eigenvector will be nonzero, from this it follows that λ is an eigenvalue of G. □

Lemma 3 shows that each eigenvector of $\hat{G}$ lifts to an eigenvector of G. Conversely every eigenvector of G pushes down to an "eigenvector" of $\hat{G}$. Suppose $\mathbf{x}$ is an eigenvector of G satisfying (1) for eigenvalue λ, $\hat{\mathbf{x}}$ is related to $\mathbf{x}$ as described in Section 2, and for $\hat{v}$ a vertex in $\hat{G}$ that $\pi^{-1}(\hat{v}) = \{s,t\}$ in G, then

$$\sum_{\hat{u}\sim\hat{v}} w(\hat{u},\hat{v})\hat{\mathbf{x}}(\hat{u}) = \frac{1}{2}\sum_{u\sim s} w(w,s)\mathbf{x}(u) + \frac{1}{2}\sum_{u\sim t} w(w,t)\mathbf{x}(u) = \lambda\big(\frac{\mathbf{x}(s)+\mathbf{x}(t)}{2}\big) = \lambda\hat{\mathbf{x}}(\hat{v}),$$

showing that the eigenvector pushes down. However, it might push down to the trivial eigenvector, i.e., $\mathbf{0}$ (the all zero vector), and it is this case which interests us now.

Lemma 4. *Suppose that G is a double edge covering of $\hat{G}$ and $\mathbf{x}$ is an eigenvector of G orthogonal to the span of the lift of the eigenvectors of $\hat{G}$. If $\pi(s) = \pi(t)$ and $s \neq t$ then $\mathbf{x}(s) = -\mathbf{x}(t)$. In particular, the eigenvector pushes to $\mathbf{0}$.*

Proof. Let $\mathbf{x}$ be an eigenvector of G orthogonal to the span of the lift of the eigenvectors of $\hat{G}$, and push the eigenvector down to $\hat{G}$ and then lift it up to form $\mathbf{x}'$. By construction $\mathbf{x}'$ is in the span of the lift of eigenvectors of $\hat{G}$ and so orthogonal to $\mathbf{x}$.

Further, by the pushing then lifting we have that $\mathbf{x}'(s) = \mathbf{x}'(t) = \frac{1}{2}\big(\mathbf{x}(s) + \mathbf{x}(t)\big)$. We then have

$$0 = \mathbf{x}^*\mathbf{x}' = \sum \overline{\mathbf{x}(s)}\mathbf{x}'(s) = \sum \overline{\mathbf{x}(s)}\frac{1}{2}\big(\mathbf{x}(s) + \mathbf{x}(t)\big) = \frac{1}{2}\sum_{\substack{s\neq t\\ \pi(s)=\pi(t)}} \big|\mathbf{x}(s) + \mathbf{x}(t)\big|^2,$$

from which it follows that $\mathbf{x}(s) = -\mathbf{x}(t)$ for all $\pi(s) = \pi(t)$ with $s \neq t$. $\square$

Lemma 4 shows that the remaining eigenvectors of G also have structure related to its covering of $\hat{G}$. We now fix $\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n$ to denote the vertices of $\hat{G}$ and $v_1, v_{-1}, v_2, v_{-2}, \ldots, v_n, v_{-n}$ to denote the vertices of G with the property that $\pi(v_i) = \hat{v}_{|i|}$. (Of course such a labeling is not unique as we can, for instance, interchange v_i and v_{-i}.) An example of this labeling is shown for a double covering in Figure 4.

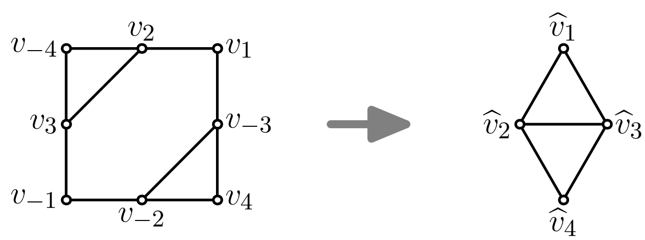

Figure 4: A labeled double covering.

Lemma 5. *Let G be a double covering of $\hat{G}$ with the vertices labeled such that $\pi(v_i) = \hat{v}_{|i|}$. Then for any i and j, $v_i{\sim}v_j$ if and only $v_{-i}{\sim}v_{-j}$.*

Proof. Let us assume that $v_i{\sim}v_j$. Then in $\hat{G}$ we have that $\hat{v}_i{\sim}\hat{v}_j$ and since $\pi(v_{-j}) = \hat{v}_j$ it follows by property (ii) of coverings there is some k so that $v_k{\sim}v_{-j}$ and $\pi(v_k) = \hat{v}_i$. Thus $v_k = v_i$ or v_{-i}, but the former case is impossible by Lemma 2, therefore we can conclude that $v_{-i}{\sim}v_{-j}$. $\square$

This symmetry allows us to condense the information about the remaining eigenvalues into another graph, which we will call the *anti-cover* of G and denote it by $\check{G}$. The anti-cover will be similar to $\hat{G}$, the major difference now being that we will allow for *negative* weights. The vertex set of $\check{G}$ is $\check{v}_1, \check{v}_2, \ldots, \check{v}_n$, with weight function

$$w(\check{v}_i, \check{v}_j) = \begin{cases} w(v_i, v_j) & \text{if } v_i{\sim}v_j \text{ in } G;\\ -w(v_{-i}, v_j) & \text{if } v_{-i}{\sim}v_j \text{ in } G;\\ 0 & \text{otherwise.} \end{cases}$$

An application of Lemma 2 shows that this is well defined and Lemma 5 shows that the definition for the weight function will be symmetric. It is also easy to observe that $\left|w(\check{v}_i,\check{v}_j)\right| = w(\hat{v}_i,\hat{v}_j)$. As an example the anti-cover graph of the double covering shown in Figure 4 is shown in Figure 5

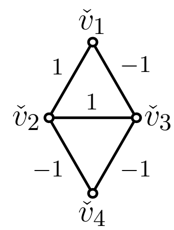

Figure 5: The anti-cover of the double covering shown in Figure 4.

The advantage of allowing these negative weights is that the eigenvectors and eigenvalues that are not found by the lifting of $\hat{G}$ can be found by the lifting of $\check{G}$. To do this we push and lift from G to $\check{G}$ by letting $\mathbf{x}(v_i) = -\mathbf{x}(v_{-i}) = \check{\mathbf{x}}(v_i)$. It then follows from Lemmas 4 and 5 that equation (1) holds in G for a value of λ and nontrivial $\mathbf{x}$ (orthogonal to the lift of eigenvectors from $\hat{G}$) if and only if the corresponding relationship holds in $\check{G}$ for λ and the corresponding vector $\check{\mathbf{x}}$. In particular the graph $\check{G}$ gives the remaining eigenvalues and eigenvectors of G. This establishes the following result.

Theorem 6. *Let G be a double covering of $\hat{G}$ and $\check{G}$ be the corresponding anti-covering. If A, $\hat{A}$ and $\check{A}$ are the corresponding adjacency matrices of these graphs, then the spectrum of A is the union of the spectrum of $\hat{A}$ and the spectrum of $\check{A}$.*

3.1 Applications of anti-coverings

Given a graph $\hat{G}$, we cannot always determine the graph G which double covers. On the other hand given an anti-cover graph $\check{G}$ then we can always uniquely determine G (and also $\hat{G}$). First to determine $\hat{G}$ it suffices to take the absolute value of all the weights, to determine G we construct by the following procedure. Let $v_1, v_{-1}, v_2, v_{-2}, \ldots, v_n, v_{-n}$ be the vertices of G and with edge weights between vertices given by the following: if $w(\check{v}_i,\check{v}_j) > 0$ then $w(v_i,v_j) = w(v_{-i},v_{-j}) = w(\check{v}_i,\check{v}_j)$; if $w(\check{v}_i,\check{v}_j) < 0$ then $w(v_i,v_{-j}) = w(v_{-i},v_j) = -w(\check{v}_i,\check{v}_j)$; all other weights will be 0.

If we treat a real symmetric matrix A as an adjacency matrix of some anti-cover $\check{G}$ then the above shows how to construct the adjacency matrix of the original graph G; on the other hand if we let $|A|$ denote the matrix where entrywise we take the absolute value then $|A|$ is the adjacency matrix of the graph $\hat{G}$. This establishes Theorem 1.

As another example of Theorem 6 we have the following.

Corollary 7. *Let A be the following $n \times n$ matrix*

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & -1 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Then the eigenvalues of A are $2\cos\big(\pi(2j+1)/n\big)$ for $j = 0, 1, \ldots, n-1$.

Proof. It is easy to check that A is the adjacency matrix of an anti-covering of the $2n$-cycle where we have wrapped the cycle on itself twice (an example for $n = 3$ is shown on the left in Figure 3). Therefore by Theorem 6 it follows that the spectrum of A is the spectrum of the $2n$ cycle with the spectrum of the n cycle removed. Since the eigenvalues of the cycles are well known the result follows. $\square$

It is known that the only double coverings of trees are trivial coverings, i.e., two duplicate copies of the tree pushed down. This establishes the following.

Corollary 8. *Let A be a real symmetric matrix such that $|A|$ is a (weighted) adjacency matrix for some tree. Then the spectrum of A is the same as the spectrum of $|A|$.*

An alternative proof of this result is to show inductively that there is some diagonal matrix Q with entries being ± 1 such that $QAQ = |A|$. This proof is given to emphasize the approach.

4 Double edge coverings and the normalized Laplacian

We now consider the more general case of double edge coverings without the requirement that vertices of G map 2-to-1 onto $\hat{G}$. For example, a double edge covering of the six cycle covering a path of length four is shown in Figure 6. A calculation shows that there are no common eigenvalues of the adjacency matrices between the path of length four and the six cycle (as had happened in the previous section). This is caused by the folding at the vertices t_1 and t_2. However, if we consider the normalized Laplacian then the eigenvalues of the six cycle are $(2, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 0)$ while the eigenvalues of the path of length four are $(2, \frac{3}{2}, \frac{1}{2}, 0)$ so that we do have containment. This holds in general.

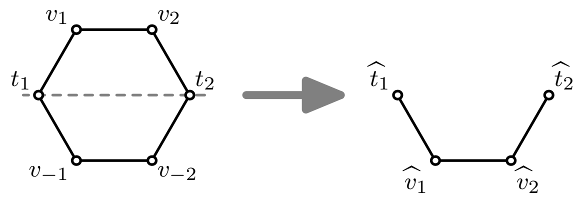

Figure 6: An example of a double edge covering which is not a double covering.

Lemma 9. *Suppose that G is a double edge covering of $\hat{G}$. Then the eigenvalues of the normalized Laplacian matrix of $\hat{G}$ (counting multiplicity) are also eigenvalues of the normalized Laplacian matrix of G.*

Proof. Let $\hat{\mathbf{x}}$ be a nonzero harmonic eigenvector of $\hat{G}$ associated with eigenvalue λ and let $\mathbf{x}$ be the lifted vector on G described at the end of Section 2. There are now two cases, depending on the number of preimages of a vertex $\hat{v}$ in $\hat{G}$. First suppose $\hat{v}$ has two preimages, then as in Lemma 3 we have a bijective correspondence between the edes incident to v and $\hat{v}$. It follows that $d(v) = d(\hat{v})$ and so we have

$$\begin{aligned} d(v)\mathbf{x}(v) - \sum_{u\sim v} w(u,v)\mathbf{x}(u) &= d(\hat{v})\hat{\mathbf{x}}(\hat{v}) - \sum_{\hat{u}\sim\hat{v}} w(\hat{u},\hat{v})\hat{\mathbf{x}}(\hat{u}) \\ &= \lambda d(\hat{v})\hat{\mathbf{x}}(\hat{v}) \;=\; \lambda d(v)\mathbf{x}(v). \end{aligned}$$

Now suppose that $\hat{v}$ has one preimage, then from Lemma 2 if $\pi(v) = \hat{v}$ then the double edge covering will "fold" the vertex v. In particular, there is a 2-to-1 correspondence between the edges incident to v and to $\hat{v}$, so that $d(v) = 2d(\hat{v})$. So we have

$$\begin{aligned} d(v)\mathbf{x}(v) - \sum_{u\sim v} w(u,v)\mathbf{x}(u) &= 2d(\hat{v})\hat{\mathbf{x}}(\hat{v}) - 2\sum_{\hat{u}\sim\hat{v}} w(\hat{u},\hat{v})\hat{\mathbf{x}}(\hat{u}) \\ &= 2\lambda d(\hat{v})\hat{\mathbf{x}}(\hat{v}) \;=\; \lambda d(v)\mathbf{x}(v). \end{aligned}$$

Combining the above shows that λ is an eigenvalues of G. $\square$

As before every harmonic eigenvector of G pushes down to a "harmonic eigenvector" of $\hat{G}$. Suppose $\mathbf{x}$ is a harmonic eigenvector of G satisfying (2) for eigenvalue λ, $\hat{\mathbf{x}}$ is related to $\mathbf{x}$ as described in Section 2. If $\hat{v}$ is a vertex in $\hat{G}$ for which $\pi^{-1}(\hat{v}) = \{s,t\}$ in G, then

$$\begin{aligned} d(\hat{v})\hat{\mathbf{x}}(\hat{v}) - \sum_{\hat{u}\sim\hat{v}} w(\hat{u},\hat{v})\hat{\mathbf{x}}(\hat{u}) &= \frac{1}{2}\Bigg(d(s)\mathbf{x}(s) - \sum_{u\sim s} w(u,s)\mathbf{x}(u) + \\ &\qquad\qquad d(t)\mathbf{x}(t) - \sum_{u\sim t} w(u,t)\mathbf{x}(u)\Bigg) \\ &= \frac{1}{2}\big(\lambda d(s)\mathbf{x}(s) + \lambda d(t)\mathbf{x}(t)\big) \;=\; \lambda d(\hat{v})\hat{\mathbf{x}}(\hat{v}). \end{aligned}$$

If on the other hand $\hat{v}$ is a vertex in $\hat{G}$ for which $\pi^{-1}(\hat{v}) = \{s\}$ in G, then

$$\begin{aligned} d(\hat{v})\hat{\mathbf{x}}(\hat{v}) - \sum_{\hat{u}\sim\hat{v}} w(\hat{u},\hat{v})\hat{\mathbf{x}}(\hat{u}) &= \frac{1}{2}d(s)\mathbf{x}(s) - \frac{1}{2}\sum_{u\sim s} w(u,s)\mathbf{x}(u) \\ &= \frac{1}{2}\lambda d(s)\mathbf{x}(s) \;=\; \lambda d(\hat{v})\hat{\mathbf{x}}(\hat{v}). \end{aligned}$$

Thus every harmonic eigenvector pushes down. But as before it might push down to $\mathbf{0}$ and it is this case that we focus on. Before stating the analog of Lemma 4 we first introduce the convention of orthogonality for two harmonic eigenvectors. Namely, we say two harmonic eigenvectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal if and only if $\mathbf{x}^*D\mathbf{y} = 0$ (i.e., if the eigenvectors to which they correspond are zero). With this convention and the symmetry of the normalized Laplacian it is easy to show that the matrix has a full set of orthonormal harmonic eigenvectors.

Lemma 10. *Suppose that* $\mathbf{x}$ *is a harmonic eigenvector of* G *orthogonal (following the above convention) to the span of the lift of the harmonic eigenvectors of* $\hat{G}$ *then if* $\pi(s) = \pi(t)$ *and* $s \neq t$ *then* $\mathbf{x}(s) = -\mathbf{x}(t)$ *while if there is no* t *for which* $s \neq t$ *and* $\pi(s) = \pi(t)$ *then* $\mathbf{x}(s) = 0$*. In particular, the eigenvector pushes to* $\mathbf{0}$*.*

The proof of Lemma 10 follows the proof of Lemma 4 and will be omitted.

As before we can use this to label the vertices of the graph G. There are now though two categories. First we will fix $\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_m$ as vertices in $\hat{G}$ with two preimages and label their preimages as $v_1, v_{-1}, v_2, v_{-2}, \ldots, v_m, v_{-m}$ with $\pi(v_i) = v_{|i|}$. Second there will be labels $\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_\ell$ as vertices in $\hat{G}$ with one preimage and label their preimages as $t_1, t_2, \ldots, t_\ell$. (An example of this labeling is shown in the double edge covering in Figure 6.)

We now turn to constructing the anti-cover $\check{G}$ for the normalized Laplacian, i.e., the graph which will have the eigenvalues not found by lifting harmonic eigenvectors of $\hat{G}$. Since the remaining harmonic eigenvectors are 0 at the t_i the contribution of $w(v_j, t_i)\mathbf{x}(t_i)$ will be 0. Thus the contribution of any edges incident to a vertex t_i will be only in the degree terms. In the anti-cover we will thus remove all of the t_i and any edges incident to t_i are contracted to the other incident vertex and will put its weight on that vertex. We will represent this additional weight at a vertex pictorially by marking a vertex as" $\boxed{k}$ ", where k is the sum of the weights of the contracted edges incident at that vertex. This additional weight at the vertices can be stored as a function of weights on the vertices $w : V \to [0, \infty)$. The edge weights for the remaining edges are then assigned as before in Section 3. As an example, the graph in Figure 3 has as its anti-cover the graph $\boxed{1}\!\!-\!\!-\!\!-\!\!\boxed{1}$.

Finally, in order to satisfy the same relationship (2) in G as in $\check{G}$ then we need to define the degree of a vertex in $\check{G}$ by

$$d(\hat{v}) = w(\hat{v}) + \sum_{\hat{u}\sim\hat{v}} \big|w(\hat{u},\hat{v})\big|.$$

We now find that the harmonic eigenvectors for our graph $\check{G}$ lift to harmonic eigenvectors of G by the rule $\mathbf{x}(v_i) = -\mathbf{x}(v_{-i}) = \check{\mathbf{x}}(\check{v}_i)$ and $\mathbf{x}(t_i) = 0$, in particular they lift to all of the harmonic eigenvectors missed by $\hat{G}$.

If we define $\check{\mathcal{L}}$ by using the above weight function for the adjacency matrix and the above definition for the degree matrix then we have established the following result.

Theorem 11. *Let G be a double edge covering of $\hat{G}$ and $\check{G}$ be the corresponding anti-covering. If $\mathcal{L}$, $\hat{\mathcal{L}}$ and $\check{\mathcal{L}}$ are the corresponding normalized Laplacian matrices of these graphs, then the spectrum of $\mathcal{L}$ is the union of the spectrums of $\hat{\mathcal{L}}$ and $\check{\mathcal{L}}$.*

4.1 Hearing the shape of an anti-cover

In Section 3.1 we saw that for the adjacency matrix that given an anti-cover we could uniquely determine the graph. For the normalized Laplacian this is no longer the case. The problem lies in that we cannot tell how many t_i there are and how they connect to the vertices with excess weight.

For example, all three of the graphs in Figure 1 have left/right symmetry which gives a natural folding and so a natural double edge covering. If we then find the anti-covering of these graphs we see that they all share the anti-cover of $\boxed{1}-\boxed{1}-\boxed{1}-\boxed{1}$. Thus there common eigenvalues come from a shared anti-cover rather than a shared covering or equitable partition.

Note that the adjacency matrix cannot always detect this type of shared anti-coverings. But this comes at a cost. A famous question in spectral graph theory is "can you hear the shape of a graph?" That is, given the eigenvalues can you determine the graph that produced them. There are many examples of two graphs that share the same spectrum but are not the same graph, while there are also examples of graphs that are uniquely determined by the spectrum.

But when it comes to the normalized Laplacian of these anti-covers the situation can be even worse. Consider the two weighted graphs in Figure 7 (with any weights not indicated being 1). These graphs share four nontrivial eigenvalues. They also have obvious left/right symmetry and the eigenvalues found by the double edge-covering when the graphs are folded in half are the ones which are not common. Thus the shared eigenvalues come from their anti-covers. These are simple to construct and are shown in Figure 8. However it is not possible to obtain one of these anti-coverings from the other by relabeling and/or scaling. So the shared eigenvalues are the result

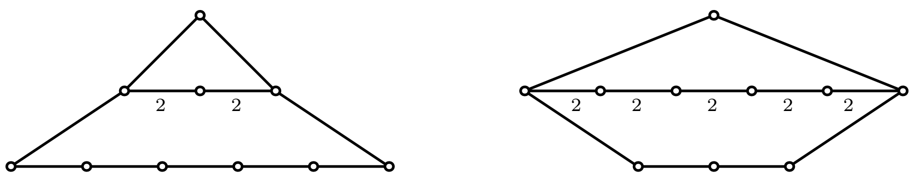

Figure 7: Two graphs with several shared eigenvalues.

Figure 8: Anti-covers of the graphs shown in Figure 7

of two cospectral anti-covers. However the situation is a little more interesting than that. If we now compute $\check{\mathcal{L}}$ for both these graphs they both give

$$\begin{pmatrix} 1 & \frac{-1}{2\sqrt{2}} & 0 & 0 \\ \frac{-1}{2\sqrt{2}} & 1 & \frac{-1}{2} & 0 \\ 0 & \frac{-1}{2} & 1 & \frac{-1}{2} \\ 0 & 0 & \frac{-1}{2} & \frac{3}{2} \end{pmatrix}.$$

In particular, the anti-cover graph cannot in general be uniquely determined from the corresponding normalized Laplacian of the anti-cover. So these graphs are not only co-spectral they are also co-normalized Laplacian.

5 Concluding comments

We have considered graphs G which double edge cover other graphs and shown how to find the spectrum of G by finding the spectrum of two smaller graphs. There are still some avenues open for investigation on this problem. For instance how far can the analysis in Section 3 be carried over to eigenvalues of directed graphs? (The similar problem for Section 4 is more problematic as there is often no well defined normalized Laplacian for a directed graph.) Perhaps the most famous theorem about nonnegative matrices is the Peron-Frobenius Theorem. Can the construction used to prove Theorem 1 be used to extend any results from nonnegative matrices to real matrices?

Is there an analogous result for coverings which are triple edge coverings, quadruple edge coverings, or more generally k-edge coverings? (In this direction we remark that chemists have used a technique similar to the one presented here to find the eigenvalues of graphs with three-fold rotational symmetry.) We remark here that analogs of Lemmas 4 and 10 are now much more complex since now the sum of three or more entries is 0 which adds another layer of difficulty.

In Sections 3 and 4 we gave examples of graphs which used negative weights on the edges (and in Section 4 also used a weight function on the vertices). It would be interesting to see if these graphs can be used in other contexts and how far a generalized version of weighted graphs (i.e., where weights are allowed to be any real number and weights can be put on the vertices) can be pushed.