

# Cospectral graphs for the normalized Laplacian

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(joint work with Jason Grout)

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# Spectral graph theory

Graphs can be associated with matrices. By studying the eigenvalues of these matrices we can understand something about the graph, different matrices can give us different information about the graph.

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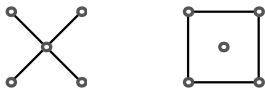
Graphs can be associated with matrices. By studying the eigenvalues of these matrices we can understand something about the graph, different matrices can give us different information about the graph.

- $A$  The adjacency matrix. 1 indicates an edge, 0 a non-edge.
- $L$  The (combinatorial) Laplacian.  $L = D - A$ , where  $D$  is diagonal degree matrix.
- $\mathcal{L}$  The normalized Laplacian.  $\mathcal{L} = D^{-1/2}LD^{-1/2}$ .

(We will assume that our graphs are simple, no loops, no multiple edges and also no isolated vertices.)

# Using spectrum to test for connectedness

- A The adjacency matrix cannot determine if a graph is connected only by the spectrum. For example the following graphs are cospectral:



- L The (combinatorial) Laplacian can determine the number of connected components, it is equal to the multiplicity of 0 as an eigenvalue.
- L The normalized Laplacian can determine the number of connected components, it is equal to the multiplicity of 0 as an eigenvalue.

# Using spectrum to test for being bipartite

- A The adjacency matrix can determine if a graph is bipartite, namely a graph is bipartite if and only if the spectrum is symmetric around 0.
- L The (combinatorial) Laplacian cannot determine if a graph is bipartite only by the spectrum. For example the following graphs are cospectral:



- L The normalized Laplacian can determine if a graph is bipartite, namely a graph is bipartite if and only if the spectrum is symmetric around 1.

# Using spectrum to count edges

- $A$  The adjacency matrix can determine the number of edges, namely it is half of the sum of squares of the eigenvalues.
- $L$  The (combinatorial) Laplacian can determine the number of edges, namely it is the sum of the eigenvalues.
- $\mathcal{L}$  The normalized Laplacian cannot determine the number of edges only by the spectrum. For example the following graphs are cospectral:



# Finding cospectral graphs

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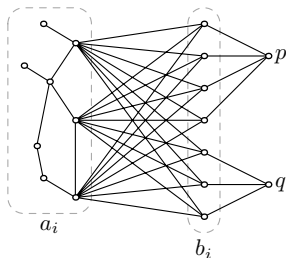
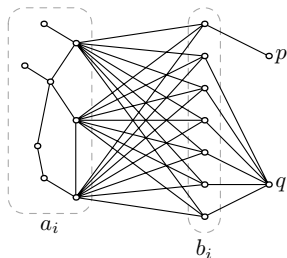
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# Edge swapping

Let  $G$  and  $H$  be the graphs of the type shown below, where for  $G$  we have  $p$  is adjacent to  $b_1, \dots, b_k$  and  $q$  is adjacent to  $b_{k+1}, \dots, b_m$ ; and for  $H$  we have  $p$  is adjacent to  $b_1, \dots, b_\ell$  and  $q$  is adjacent to  $b_{\ell+1}, \dots, b_m$ . Further the  $b_i$  all have common neighbors (except possibly  $p$  and  $q$ ).

Then  $G$  and  $H$  are cospectral for the normalized Laplacian.



# Basic tools and notation

We will use **harmonic eigenvectors** to show that the graphs are cospectral. Given an eigenvector  $\mathbf{x}$ , i.e.,  $\mathcal{L}\mathbf{x} = \lambda\mathbf{x}$ , then the harmonic eigenvector is  $\mathbf{y} = D^{-1/2}\mathbf{x}$  which becomes  $(D - A)\mathbf{y} = \lambda D\mathbf{y}$  or

$$\sum_{u \sim v} \mathbf{y}(u) = (1 - \lambda)\mathbf{y}(v)d(v).$$

Two harmonic eigenvectors  $\mathbf{y}_1$  and  $\mathbf{y}_2$  are orthogonal if  $\mathbf{y}_2^* D \mathbf{y}_1 = 0$  (i.e., the eigenvectors are orthogonal).

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We use  $\mathbf{e}[v]$  to denote the vector which is 1 at vertex  $v$  and 0 otherwise.

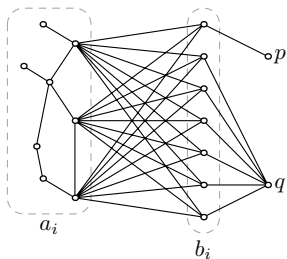
We let  $s$  denote the degree of the  $b_i$ .





# Proof (assuming the $b_i$ induce an empty graph)

$$\sum_{u \sim v} \mathbf{y}(u) = (1 - \lambda) \mathbf{y}(v) d(v).$$



For  $\lambda = 1$  we have:

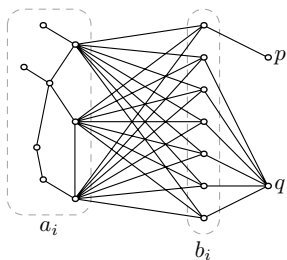
$$\begin{aligned} \mathbf{e}[b_1] - \mathbf{e}[b_i] & \quad \text{for } 2 \leq i \leq k-1 \\ \mathbf{e}[b_m] - \mathbf{e}[b_i] & \quad \text{for } k \leq i \leq m-1 \end{aligned}$$

For  $\lambda = 1 - 1/\sqrt{s}$  we have:

$$\begin{aligned} \mathbf{y}_1 = & (m - k)(\mathbf{e}[b_1] + \cdots + \mathbf{e}[b_k]) \\ & - k(\mathbf{e}[b_{k+1}] + \cdots + \mathbf{e}[b_m]) \\ & + \sqrt{s}(m - k)\mathbf{e}[p] - \sqrt{s}k\mathbf{e}[q]. \end{aligned}$$

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For  $\lambda = 1 + 1/\sqrt{s}$  we have:

$$\begin{aligned} \mathbf{y}_2 = & (m - k)(\mathbf{e}[b_1] + \cdots + \mathbf{e}[b_k]) \\ & - k(\mathbf{e}[b_{k+1}] + \cdots + \mathbf{e}[b_m]) \\ & - \sqrt{s}(m - k)\mathbf{e}[p] + \sqrt{s}k\mathbf{e}[q]. \end{aligned}$$

# Proof (assuming the $b_i$ induce an empty graph)

The remaining harmonic eigenvectors are orthogonal to:

$$\mathbf{e}[b_1] - \mathbf{e}[b_i] \quad \text{for } 2 \leq i \leq k - 1,$$

$$\mathbf{e}[b_m] - \mathbf{e}[b_i] \quad \text{for } k \leq i \leq m - 1,$$

$$\frac{1}{2}(\mathbf{y}_1 + \mathbf{y}_2) = (m - k)(\mathbf{e}[b_1] + \cdots + \mathbf{e}[b_k]) - k(\mathbf{e}[b_{k+1}] + \cdots + \mathbf{e}[b_m]),$$

$$\frac{1}{2\sqrt{s}}(\mathbf{y}_1 - \mathbf{y}_2) = (m - k)\mathbf{e}(p) - k\mathbf{e}(q).$$

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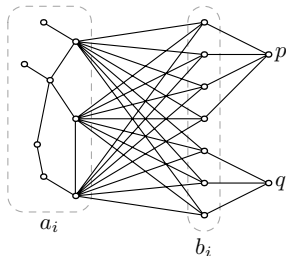
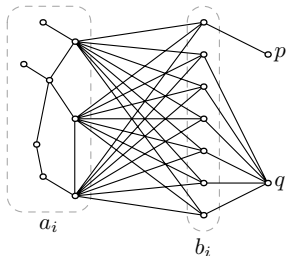
$$\frac{1}{2}(\mathbf{y}_1 + \mathbf{y}_2) = (m - k)(\mathbf{e}[b_1] + \cdots + \mathbf{e}[b_k]) - k(\mathbf{e}[b_{k+1}] + \cdots + \mathbf{e}[b_m]),$$

$$\frac{1}{2\sqrt{s}}(\mathbf{y}_1 - \mathbf{y}_2) = (m - k)\mathbf{e}(p) - k\mathbf{e}(q).$$

In particular, if  $\mathbf{z}$  is one of the remaining harmonic eigenvectors then it is **constant** on the  $b_i$ ; and  $\mathbf{z}(p) = \mathbf{z}(q)$ .

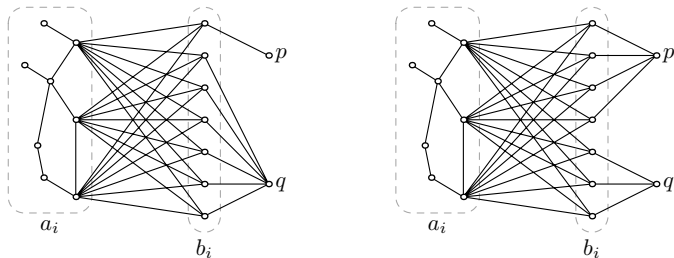
# Proof (assuming the $b_i$ induce an empty graph)

If a harmonic eigenvector is constant on the  $b_i$  and  $\mathbf{z}(p) = \mathbf{z}(q)$ , then it is a harmonic eigenvector for **both**  $G$  and  $H$ .



# Proof (assuming the $b_i$ induce an empty graph)

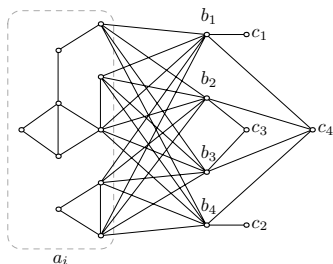
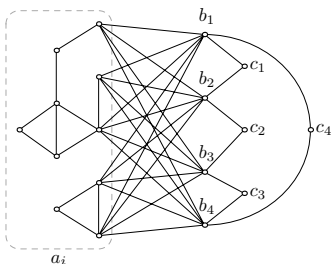
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In particular, the remaining harmonic eigenvectors are the same for both graphs, and so the graphs are cospectral.  $\square$

# Another example

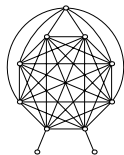
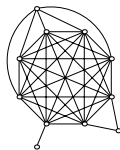
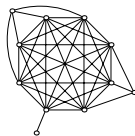
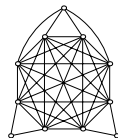
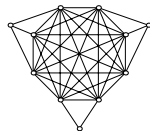
The approach that we used can establish other examples of cospectral graphs. As an example, the following are cospectral.



# A cospectral family of graphs – Fuzzy balls

We can now form large families of graphs all of which are mutually cospectral.

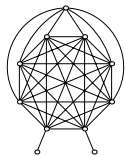
For  $m_1 + m_2 + \dots + m_k = n$ , let  $FB(m_1, m_2, \dots, m_k)$  be the graph on  $n + k$  vertices  $b_1, \dots, b_n, v_1, \dots, v_k$  where the  $b_i$  induce a complete graph, each  $v_i$  is adjacent to exactly  $m_i$  vertices and each  $b_i$  is adjacent to exactly one of the  $v_j$

 $FB(6, 1, 1)$  $FB(5, 2, 1)$  $FB(4, 3, 1)$  $FB(4, 2, 2)$  $FB(3, 3, 2)$

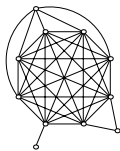
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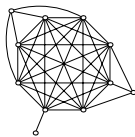
For  $m_1 + m_2 + \dots + m_k = n$ , let  $FB(m_1, m_2, \dots, m_k)$  be the graph on  $n + k$  vertices  $b_1, \dots, b_n, v_1, \dots, v_k$  where the  $b_i$  induce a complete graph, each  $v_i$  is adjacent to exactly  $m_i$  vertices and each  $b_i$  is adjacent to exactly one of the  $v_j$



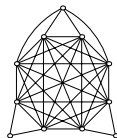
$FB(6, 1, 1)$



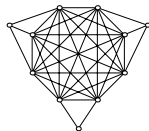
$FB(5, 2, 1)$



$FB(4, 3, 1)$



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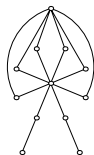
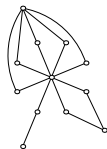
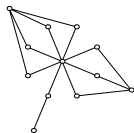
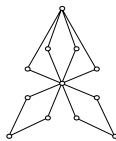
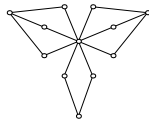


$FB(3, 3, 2)$

By swapping  $FB(m_1, m_2, \dots, m_k)$  is cospectral with  $FB(m_1 + m_i - 1, m_2, \dots, m_{i-1}, 1, m_{i+1}, \dots, m_k)$ .

# A cospectral family of graphs – Inflated stars

For  $m_1 + m_2 + \dots + m_k = n$ , let  $IS(m_1, m_2, \dots, m_k)$  be the graph on  $n + k + 1$  vertices  $a, b_1, \dots, b_n, v_1, \dots, v_k$  where  $a$  and the  $b_i$  induce a star graph with  $a$  as the central vertex, each  $v_i$  is adjacent to exactly  $m_i$  vertices and each  $b_i$  is adjacent to exactly one of the  $v_j$ .

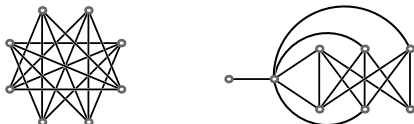

 $IS(6, 1, 1)$ 

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By swapping  $IS(m_1, m_2, \dots, m_k)$  is cospectral with  $IS(m_1 + m_i - 1, m_2, \dots, m_{i-1}, 1, m_{i+1}, \dots, m_k)$ .

# Comments about the spectrum of the normalized Laplacian

Edge swapping was discovered by looking at cospectral graphs for graphs with  $\leq 8$  vertices, generated using SAGE.  
There are other cospectral graphs not related by swapping.

- The spectrum of  $\mathcal{L}$  cannot detect if a graph is regular.



- A graph and a subgraph can be cospectral for  $\mathcal{L}$ .



Thank you!