An Erdős-Ko-Rado problem on the strip

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Extremal set theory

We will consider families of sets, $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$, where we usually assume each $A_i \subseteq [n]$. Extremal set theory problems ask given some constraint on the properties of the sets what is the maximum size of the family.

Sperner 1928

If $A_i \not\subseteq A_j$ for all $i \neq j$ (in other words the family forms an anti-chain) then

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

Erdős-Ko-Rado Theorem

A family of sets $\mathcal{F}$ is said to be intersecting if the intersection of any two elements in $\mathcal{F}$ is nonempty.

Erdős-Ko-Rado 1961

Let $\mathcal{F}$ be a family of k element sets which is intersecting. If $2k \leq n$ then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

- Original proof used induction. Katona 1972 gave a double counting proof, which is now the best known proof.
- If $2k > n$ then all sets must trivially intersect by the pigeon hole principle.
- To construct a family achieving this maximal size we can fix a single element of $[n]$ and then consider all k element sets which contain that element.

More general Erdős-Ko-Rado Theorem

Let $\mathcal{F}$ be a family of k element sets, any two of which intersect in at least ℓ elements. If $n \geq (k - \ell + 1)(\ell + 1)$ then

$$|F| \leq \binom{n-\ell}{k-\ell}.$$

- Result was originally known to hold for $n \geq n_0(k, \ell)$. Frankl 1978 established the bound for $\ell \geq 15$ and Wilson (1984) established the bound in general.

Erdős-Ko-Rado flavored problems

The Erdős-Ko-Rado problem deals with intersections of sets. This has inspired similar investigations into other intersection problems. What is needed is (1) a combinatorial object which has (2) some notion of intersection.

- (Deza-Frankl 1977; Cameron-Ku 2003) Permutations, where two permutations π and σ intersect if $\pi(i) = \sigma(i)$ for some i.
- (Ku-Renshaw 2008) Permutations, where two permutations π and σ intersect if they share a common cycle in their cycle decompositions.
- (Ku-Renshaw 2008) Set partitions, where two set partitions intersect if they share a block in common.

More Erdős-Ko-Rado flavored problems

- (Bollobás-Leader 1997) Colored sets, where two colored sets intersect if there is a common element to both sets which has the same color in both sets.
- (Ford 1999) Arithmetic progressions of real numbers, where two arithmetic progressions are intersecting when they share t elements.
- (Frankl-Füredi 1980; Frankl-Tokushige 1999) Integer sequences, $(a_1, \ldots, a_n)$ with $1 \leq a_i \leq q$ and intersection corresponds to agreement in entries.
- (Frankl-Wilson 1986) Vector spaces, where two vector spaces are t-intersecting if the dimension of their intersection is of size t.

The n-strip

Our underlying object can be thought of as a strip of paper with n squares labelled $1, 2, \ldots, n$: $\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & \cdots & n \\ \hline\end{array}$

Our base family of objects are the different ways to *decompose* the strip into k parts (i.e., tear the paper into k pieces).

Example

Some decompositions for $n = 6$ and $k = 3$:

$$A = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}\quad \begin{array}{|c|}\hline 4 \\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 5 & 6 \\ \hline\end{array}$$

$$B = \begin{array}{|c|}\hline 1 \\ \hline\end{array}\quad \begin{array}{|c|}\hline 2 \\ \hline\end{array}\quad \begin{array}{|c|c|c|c|}\hline 3 & 4 & 5 & 6 \\ \hline\end{array}$$

$$C = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 3 & 4 \\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 5 & 6 \\ \hline\end{array}$$

Intersection

We will say that two different decompositions intersect if they share a common (labelled) substrip.

Example

Some decompositions for $n = 6$ and $k = 3$:

$$\begin{aligned} A &= \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \quad \begin{array}{|c|}\hline 4\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 5&6\\\hline\end{array} \\ B &= \begin{array}{|c|}\hline 1\\\hline\end{array} \quad \begin{array}{|c|}\hline 2\\\hline\end{array} \quad \begin{array}{|c|c|c|c|}\hline 3&4&5&6\\\hline\end{array} \\ C &= \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 3&4\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 5&6\\\hline\end{array} \end{aligned}$$

$$A \cap B = B \cap C = \emptyset \qquad A \cap C = \begin{array}{|c|c|}\hline 5&6\\\hline\end{array}$$

More generally, the size of an intersection is the number of common labelled substrips.

Alternative interpretation: ordered partitions

There is a natural pairing of decompositions of the n-strip into k parts and ordered partitions of n into k parts, i.e., $n = a_1 + a_2 + \cdots + a_k$. Namely, by using the size of the substrips to be the a_k.

Example

$$\boxed{1}\quad \boxed{2\,|\,3\,|\,4}\quad \boxed{5\,|\,6}\quad \boxed{7\,|\,8\,|\,9\,|\,10}\quad \boxed{11}$$

$$\iff \quad 1+3+2+4+1.$$

Then we say $a_1 + a_2 + \cdots + a_k$ intersect $b_1 + b_2 + \cdots + b_k$ if for some i and j we have

(1) $a_1 + \cdots + a_{i-1} = b_1 + \cdots + b_{j-1}$; and

(2) $a_i = b_j$.

Alternative interpretation: subsets of $\mathbb{Z}_n$

There is another natural pairing of decompositions of the n-strip into k parts and k-sets of $\mathbb{Z}_n$. Namely, given such a decomposition record the value of the last square in every substrip.

Example

$$\boxed{1}\quad\boxed{2\,|\,3\,|\,4}\quad\boxed{5\,|\,6}\quad\boxed{7\,|\,8\,|\,9\,|\,10}\quad\boxed{11}$$

$$\iff \quad \{1, 4, 6, 10, 11\}.$$

(Note that $b_k = n$.)

Then we say $\{a_1, a_2, \ldots, a_k\}$ and $\{b_1, b_2, \ldots, b_k\}$ intersect if either $a_1 = b_1$ or for some i and j, $a_i = b_j$ and $a_{i+1} = b_{j+1}$. In other words they share two consecutive elements of $\mathbb{Z}_n$.

Lemma

The number of decompositions of the n-strip into k parts is

$$\binom{n-1}{k-1}.$$

Proof: We have to choose $k-1$ different places out of $n-1$ possible places to break the strip.

Lemma

There is a family $\mathcal{F}$ of decompositions of the n-strip into k parts which is intersecting and

$$|\mathcal{F}| = \binom{n-2}{k-2}.$$

Proof: Take all the decompositions that start with $\boxed{1}$. There remain $k-2$ different places left to choose the breaks out of a possible $n-2$.

Main question

What is the maximal size of an intersecting family $\mathcal{F}$ of decompositions of the *n*-strip into *k* parts?

By the previous slide we know that such a maximal family will satisfy

$$\binom{n-2}{k-2} \leq |\mathcal{F}| \leq \binom{n-1}{k-1}.$$

The case for k “large”

Theorem

If $k > \frac{3}{4}n$ then all decompositions of the n-strip into k parts intersect.

Proof: Given n and k it is easy to check there are at least $2k - n$ substrips of size 1. So for $k > \frac{3}{4}n$ each decomposition has more than $2\left(\frac{3}{4}n\right) - n = \frac{1}{2}n$ substrips of length 1. So by the pigeon hole principle, for any two decompositions there must be a common block of size 1, i.e., they intersect.

For $k = \frac{3}{4}n$

Theorem

If $k = \frac{3}{4}n$, and $m = \frac{1}{3}k = \frac{1}{4}n$ then the maximal size of an intersecting family on the n-strip decomposed into k parts is

$$\frac{3}{4}\binom{4m}{3m} - \frac{1}{2}\binom{2m}{m}.$$

Further there are $\sqrt{2}^{\binom{2m}{m}}$ possible maximal families.

Observation: Any decomposition in this case must have at least $2m$ substrips of size 1. If a decomposition has more than $2m$ substrips of size 1 it must automatically intersect every other such decomposition.

For $k = \frac{3}{4}n$ (continued)

The only way for two decompositions A and B to *not* intersect is if they both have $2m$ parts of size 1 and m parts of size 2. Moreover, the occurrences of parts of size 1 in one decomposition must occur opposite parts of size 2 in the other.

$$\boxed{i}\ \boxed{i+1} \in A \quad \Leftrightarrow \quad \boxed{i \,|\, i+1} \in B$$

$$\boxed{i \,|\, i+1} \in A \quad \Leftrightarrow \quad \boxed{i}\ \boxed{i+1} \in B$$

So there are $\frac{1}{2}\binom{2m}{m}$ nonintersecting pairs. A maximal family is formed by taking one element from each nonintersecting pair in addition to all other decompositions.

The case for *k* “small”

For $k = 1$ we have obvious bound of 1.

Theorem

If $2 \leq k \leq \frac{1}{3}n + 1$ then the maximal size of an intersecting family is

$$\binom{n-2}{k-2}.$$

Proof: Using the interpretation as subsets of $\mathbb{Z}_n$ such an intersecting family corresponds to sets with two *consecutive* elements in common between any two sets. This forms a system where any two sets have two elements in common. By the Erdős-Ko-Rado Theorem we know that for $n \geq 3(k-1)$ that the maximal size of such a family is bounded by $\binom{n-2}{k-2}$. Since we can achieve this bound, the result now follows.

Bounds for k in the "middle"

A "good" candidate for maximal family

$$\mathcal{F} = \{\text{all decompositions starting with } \boxed{1}\} \bigcup \{\text{all "universally intersecting" decompositions}\}$$

The obvious candidates for being universally intersecting are decompositions with lots of blocks of size 1. In particular if $(k-1)+(2k-n) > n$, or $k > \frac{1}{3}(2n+1)$, such decompositions exist by pigeon hole principle.

Theorem

For $\frac{1}{3}(2n+1) < k < \frac{3}{4}n$ there is an intersecting family $\mathcal{F}$ so that

$$|\mathcal{F}| = \binom{n-2}{k-2} + \sum_{\ell > 2(n-k)} \binom{k-1}{\ell}\binom{n-k-1}{k-\ell-1}.$$

Some other universal intersectors

It is possible to be universally intersecting and not be forced to by the pigeon hole principle.

Examples for $n = 11$ and $k = 8$

$\boxed{1\;2}\ \boxed{3}\ \boxed{4}\ \boxed{5}\ \boxed{6}\ \boxed{7}\ \boxed{8\;9\;10}\ \boxed{11}$

$\boxed{1\;2}\ \boxed{3}\ \boxed{4}\ \boxed{5}\ \boxed{6\;7\;8}\ \boxed{9}\ \boxed{10}\ \boxed{11}$

$\boxed{1\;2}\ \boxed{3}\ \boxed{4\;5\;6}\ \boxed{7}\ \boxed{8}\ \boxed{9}\ \boxed{10}\ \boxed{11}$

$\boxed{1\;2\;3}\ \boxed{4}\ \boxed{5}\ \boxed{6}\ \boxed{7}\ \boxed{8}\ \boxed{9\;10}\ \boxed{11}$

$\boxed{1\;2\;3}\ \boxed{4}\ \boxed{5}\ \boxed{6}\ \boxed{7\;8}\ \boxed{9}\ \boxed{10}\ \boxed{11}$

$\boxed{1\;2\;3}\ \boxed{4}\ \boxed{5\;6}\ \boxed{7}\ \boxed{8}\ \boxed{9}\ \boxed{10}\ \boxed{11}$

Summary of bounds (for maximal family)

$$\begin{cases} = 1 & \text{if } k = 1; \\ = \binom{n-2}{k-2} & \text{if } 1 < k \leq \frac{1}{3}n + 1; \\ \geq \binom{n-2}{k-2} & \text{if } \frac{1}{3}n + 1 < k \leq \frac{2}{3}n + \frac{1}{3}; \\ \geq \binom{n-2}{k-2} + \sum_{\ell > 2(n-k)} \binom{k-1}{\ell}\binom{n-k-1}{k-\ell-1} & \text{if } \frac{2}{3}n + \frac{1}{3} < k < \frac{3}{4}n; \\ = \frac{3}{4}\binom{4m}{3m} - \frac{1}{2}\binom{2m}{m} & \text{if } \frac{1}{3}k = \frac{1}{4}n = m; \\ = \binom{n-1}{k-1} & \text{if } k > \frac{3}{4}n. \end{cases}$$

Size of maximal family for small cases

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$n=1$	1						
$n=2$	1	1					
$n=3$	1	1	1				
$n=4$	1	1	2	1			
$n=5$	1	1	3	4	1		
$n=6$	1	1	4	6	5	1	
$n=7$	1	1	5	10	10	6	1

Method of computing size of maximal family is to group decompositions so that:

(1) taking one element from each group forms an intersecting family; and

(2) no two elements in any group are intersecting.

Example $n = 5$ and $k = 3$

Group the $\binom{4}{2} = 6$ decompositions as follows:

$$
\begin{aligned}
&\left\{\ \boxed{1}\ \ \boxed{2}\ \ \boxed{3}\!\boxed{4}\!\boxed{5}\ ,\ \boxed{1}\!\boxed{2}\ \ \boxed{3}\!\boxed{4}\ \ \boxed{5}\ \right\}\\
&\left\{\ \boxed{1}\ \ \boxed{2}\!\boxed{3}\!\boxed{4}\ \ \boxed{5}\ ,\ \boxed{1}\!\boxed{2}\ \ \boxed{3}\ \ \boxed{4}\!\boxed{5}\ \right\}\\
&\left\{\ \boxed{1}\ \ \boxed{2}\!\boxed{3}\ \ \boxed{4}\!\boxed{5}\ ,\ \boxed{1}\!\boxed{2}\!\boxed{3}\ \ \boxed{4}\ \ \boxed{5}\ \right\}
\end{aligned}
$$

Since we can form an intersecting family of size 3 but none of size 4 then the maximal size of an intersecting family in this case must be 3.

Open problems

- Given *n* and *k* what is the size of a maximal family?

  Conjecture: A maximal family is all decompositions starting with $\boxed{1}$ together will all universally intersecting decompositions.
- When does the trivial lower bound of $\binom{n-2}{k-2}$ hold?

  Conjecture: $k \leq \frac{1}{3}(2n+1)$.
- What if we consider intersecting families where each pair of decompositions share ℓ terms in common?
- What are some additional Erdős-Ko-Rado problems that can be explored?

THANK YOU