

Hypercube orientations with only two in-degrees

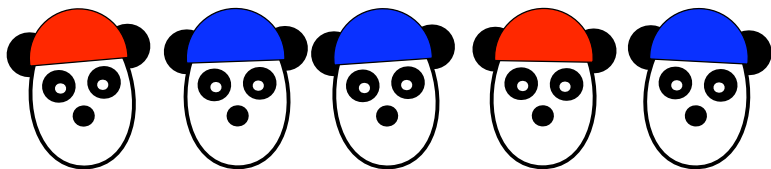
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2011 Joint Mathematics Meetings
7 January 2011

Hat guessing games



- There are n players, on each player a red or blue hat will be placed on their head.
- Each player can see all of the hats except for their own.
- After seeing what other players are wearing each player then must guess (**simultaneously** and **independently**) what kind of hat they have.
- Before the hats are placed the players are allowed to communicate and form a strategy, usually with some **objective** in mind.

Goal of having only a or b correct guesses

At the 2010 **Gathering 4 Gardner**, Iwasawa asked when it was possible for the players to form a strategy so that **no matter what hats are placed** the number of correct guesses will be either a or b .

The easy case of when $a + b = n$

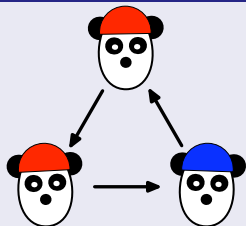
- Have a of the players guess as though the number of **blue** hats should be an **even** number.
- Have b of the players guess as though the number of **blue** hats should be an **odd** number.

Exactly one of these two groups is correct and so the number of correct guesses for this strategy is either a or b .

Also possible for $a + b \neq n$

Three players with $a = 1$ and $b = 3$

- The first player guesses the hat seen on the second player.
- The second player guesses the hat seen on the third player.
- The third player guesses the hat seen on the first player.



Not every a and b is possible. For example when there are $n = 3$ players, **no strategy** will produce either 2 or 3 correct guesses. This is because no strategy can change the expected number of correct guesses which is $n/2$.

Necessary conditions on a and b

- $0 \leq a, b \leq n.$

s = number of placements with a correct answers

t = number of placements with b correct answers

$$s + t = 2^n \quad (\text{total number of placements})$$

$$as + bt = n2^{n-1} \quad (\text{total number of correct guesses})$$

- $s = \frac{2^{n-1}(2b - n)}{b - a}$ and $t = \frac{2^{n-1}(n - 2a)}{b - a}$ must be nonnegative integers.

Main Theorem

These two necessary conditions are also sufficient for a strategy to exist.

The first few possible pairs of a and b

$n = 1$: {0, 1}
$n = 2$: {0, 2}
$n = 3$: {0, 2}, {0, 3}, {1, 2}, {1, 3}
$n = 4$: {0, 4}, {1, 3}
$n = 5$: {0, 4}, {0, 5}, {1, 3}, {1, 4}, {1, 5}, {2, 3}, {2, 4}
\vdots	\vdots

By the time we get to $n = 1000$ there are 3038 possible pairs!

By showing how we can use one strategy to make another we can reduce the number of cases that we need to consider.

Expanding our strategies

Lemma

Given a strategy for n players with either a or b correct guesses then we can also form the following strategies:

- 1 A strategy for n players with $n - a$ or $n - b$ correct guesses.
- 2 A strategy for $n + 2$ players with $a + 1$ or $b + 1$ correct guesses.
- 3 A strategy for kn players with ka or kb correct guesses.

Proof of 2

Take two players and pair them up as follows:

- The first player guesses the hat seen on the second player.
- The second player guesses the hat opposite on the first player.

The remaining n players then use the same strategy as before.

Expanding our strategies

Lemma

Given a strategy for n players with either a or b correct guesses then we can also form the following strategies:

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Corollary

We only need to generate strategies for n odd with $a = n - 2^k$ and $b = n$, where $2^k < n < 2^{k+1}$.

$$n = 1 : a = 0 \text{ and } b = 1$$

$$n = 3 : a = 1 \text{ and } b = 3$$

$$n = 5 : a = 1 \text{ and } b = 5$$

$$n = 7 : a = 3 \text{ and } b = 7$$

$$n = 9 : a = 1 \text{ and } b = 9$$

Describing a strategy

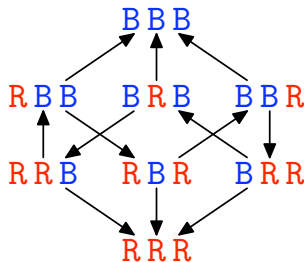
Example – as seen from the third player

The third player then knows that they are in one of two possible placements:
(red,blue,red) or (red,blue,blue).



Listing all of the possible placements of hats as **vertices**, we can indicate the decisions that have to be made as **edges** between two vertices that differ in exactly one entry.

We indicate the decisions by orienting the edges towards guesses.



deterministic strategies \longleftrightarrow hypercube orientations

Using Hamming codes

Hamming codes

A **Hamming code** is a subset of disjoint vertices of a hypercube so that each vertex is either one of these vertices or is adjacent to one of these vertices.

Hamming codes only exist for cubes with dimension $2^k - 1$.

The case when $n = 2^k - 1$, $a = 2^{k-1} - 1$ and $b = 2^k - 1$

- Starting with the n -cube select a subset of vertices H that form a Hamming code. Make all vertices of H sinks.
- Now all vertices not in H are incident to $2^k - 2$ unoriented edges. Using Euler cycles we can orient the remaining edges so that these vertices will have in-degree $2^{k-1} - 1$.

Using thickened Hamming codes

For general n to find orientations for the special cases we “thicken” up Hamming codes.

If $2^k < n < 2^{k+1}$ then we divide the strings used to describe vertices into two parts: (p, v) .

- p is a prefix string of length $a + 1 = n - 2^k + 1$.
- v is a string of length $2^k - 1$, i.e., strings for which we can find and use a Hamming code on.

To finish off this general case we do the following:

- 1 Place sinks throughout the cube.
- 2 Many other edges are then forced for some vertices to have in-degree a .
- 3 Finally, looking at all the remaining undirected edges we have that each remaining vertex is incident to exactly $2a$ of them and so we use Euler cycles to orient them.