Eigenvalues of 2-edge-coverings

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Abstract

A 2-edge-covering between G and H is an onto homomorphism from the vertices of G to the vertices of H so that each edge is covered twice and edges in H can be lifted back to edges in G. In this note we show how to compute the spectrum of G by computing the spectrum of two smaller graphs, namely a (modified) form of the covered graph H and another graph which we term the anti-cover. This is done for both the adjacency matrix and the normalized Laplacian. We also give an example of two anti-cover graphs which have the same normalized Laplacian, and state a generalization for directed graphs.

1 Introduction

Spectral graph theory looks at the connections between the structure of a graph and the eigenvalues of various matrices associated with the graph. By examining these eigenvalues (or the spectrum) some properties of a graph can be determined. When two graphs have several eigenvalues in common it can often be traced to some shared structure. One common example of this is when two graphs cover the same graph.

If we look at the spectrum of the three graphs in Figure 1 (either using the adjacency matrix or the normalized Laplacian) we see that there are four eigenvalues which are common to all three graphs. Also note that the three graphs have obvious left/right symmetry, and so we can fold each in half. In this note we will show how to compute the spectrum of graphs of this type by computing the spectrum of two smaller graphs, which we will call the (modified) cover and the anti-cover. In the example of these three graphs we will see that the common eigenvalues are traced to a shared anti-cover graph.

We will proceed in the remainder of the introduction to give a notion of a 2-edge-covering and some basic properties as well as define the matrices that we will be interested in. In Section 2 we will show how to compute the eigenvalues of the adjacency matrix for a graph which 2-edge-covers another graph, while in Section 3

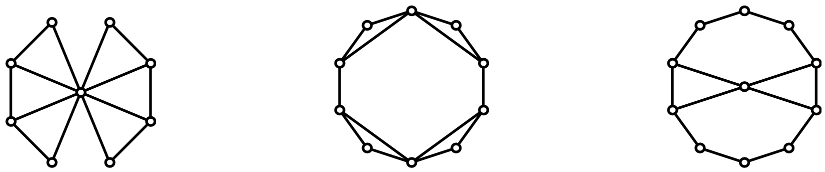

Figure 1: Three graphs sharing some common eigenvalues.

we will give a similar construction related to the normalized Laplacian. Finally, in Section 4 we will give some concluding remarks.

In this paper we will be dealing with weighted graphs. A weighted graph is a graph with a weight function on its edges $w : V \times V \to \mathbf{R}$ (usually restricted to the nonnegatives) such that $w(u, v) = w(v, u)$ (i.e., the graph is undirected) and by convention $u \sim v$ (u is adjacent to v) if and only if $w(u, v) \neq 0$. [A simple graph relates to the case when $w(u, v) \in \{0, 1\}$ for all pairs u and v.]

1.1 2-edge-coverings

We say that a graph G is a 2-edge-covering of a graph $\hat{G}$ if there is an onto map $\pi : V(G) \to V(\hat{G})$ satisfying the following conditions:

(i) if $u \sim v$ in G then $\pi(u) \sim \pi(v)$ in $\hat{G}$ and further $w\big(\pi(u), \pi(v)\big) = w(u, v)$;

(ii) if $\pi(u) \sim \hat{w}$ in $\hat{G}$ then there is some vertex v in G so that $u \sim v$ and that $\pi(v) = \hat{w}$;

(iii) for each (ordered) edge $(\hat{w}, \hat{z})$ in $\hat{G}$ there are exactly two (ordered) edges (p, q), (r, s) in G so that $\big(\pi(p), \pi(q)\big) = \big(\pi(r), \pi(s)\big) = (\hat{w}, \hat{z})$.

Property (i) states that the graph is a weight preserving homomorphism (for more about graph homomorphisms and coverings the reader is referred to Godsil and Royle [7]). Property (ii) insures that we can lift edges from $\hat{G}$ back up to G, while property (iii) states that each edge is covered twice. The reason that we insist on having ordered pairs is to deal with the creation of loops, namely, if $u \neq v$, $u \sim v$ and $\pi(u) = \pi(v)$ then we would have a loop at $\pi(u)$ in $\hat{G}$, by our convention the loop is double covered by (u, v) and (v, u).

Some examples of 2-edge-coverings involving the six cycle are shown in Figure 2 (the labeling indicating how the vertices map).

For our purposes, the most important feature of a 2-edge-covering is what happens at the vertices. It is easy to see that for a connected graph each vertex in the covered graph $\hat{G}$ can have either 1 or 2 preimages.

Lemma 1. *Let G be a nonempty connected graph which is a 2-edge-covering of $\hat{G}$ under the map π, and $\pi(v) = \hat{v}$.*

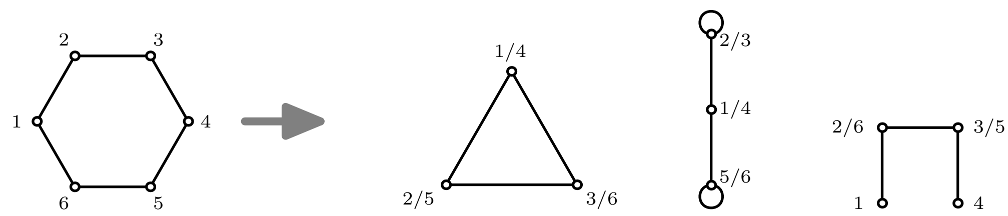

Figure 2: Examples of 2-edge-coverings involving the six-cycle.

- *If* $\left|\pi^{-1}(\hat{v})\right| = 2$ *and* $u{\sim}v{\sim}w$ *with* $u \neq w$ *in* G*, then* $\pi(u) \neq \pi(w)$*.*
- *If* $\left|\pi^{-1}(\hat{v})\right| = 1$ *and* $u{\sim}v$*, then there is some* w *such that* $v{\sim}w$*,* $u \neq w$ *and* $\pi(u) = \pi(w)$*.*

In other words when a vertex has two preimages there is a 1-to-1 correspondence between the edges incident to v and $\hat{v}$. While if a vertex has only a single preimage the edges incident to v in G map 2-to-1 to the edges incident to $\hat{v}$ in $\hat{G}$. Intuitively in the latter case the edges incident to v "fold" over and we will refer to such vertices in either G or $\hat{G}$ as a folding vertex throughout.

Proof. First consider the case $\pi^{-1}(\hat{v}) = \{v, v'\}$. Suppose that $u{\sim}v{\sim}w$, then by property (i) $\pi(u){\sim}\hat{v}$ so by property (ii) there exists some vertex z in G so that (z, v') and (u, v) are distinct edges in G both covering the same edge in $\hat{G}$. Now if $\pi(u) = \pi(w)$ then (w, v) would be a third edge in G which also covers but this contradicts (iii). Therefore we have that $\pi(u) \neq \pi(w)$.

Now consider the case $\pi^{-1}(\hat{v}) = \{v\}$. If $u{\sim}v$ then by property (i) $\pi(u){\sim}\hat{v}$. By property (iii) this edge is double covered and since $\hat{v}$ has only one preimage the two edges in G which double cover it are (u, v) and (w, v) (for some w). But now note that $v{\sim}w$, $u \neq w$ and that $\pi(u) = \pi(w)$ as needed. □

Similarly one can show that it is not possible for two folding vertices to be adjacent and we will implicitly assume this in our proofs.

1.2 Matrices of graphs and eigenvalues

There are many ways given a (weighted) graph to construct a corresponding matrix. We will be concerned with two specific matrices. The first is the adjacency matrix A and is defined entrywise by $A_{u,v} = w(u, v)$. The second is the normalized Laplacian defined by $\mathcal{L} = D^{-1/2}(D - A)D^{-1/2}$ where S is the diagonal degree matrix, i.e., $D_{u,u} = d(u) = \sum_{v\sim u} w(v, u)$. More information about the normalized Laplacian and its applications can be found in Chung [3].

Since we will be examining the eigenvalues of these matrices it is also important to consider how eigenvectors and eigenvalues relate to the graph. For the adjacency matrix A, if $\mathbf{x}$ is an eigenvector then the relationship $A\mathbf{x} = \lambda\mathbf{x}$ translates into the condition that at each vertex v

$$\sum_{u\sim v} w(u,v)\mathbf{x}(u) = \lambda\mathbf{x}(v). \tag{1}$$

For the normalized Laplacian instead of focusing on the eigenvectors we will focus on the *harmonic* eigenvectors. Namely if $\mathbf{y}$ is an eigenvector of $\mathcal{L}$ associated with λ then the harmonic eigenvector is $\mathbf{x} = D^{-1/2}\mathbf{y}$. The relationship $\mathcal{L}\mathbf{y} = \lambda\mathbf{y}$ then becomes (after some simplifying) $(D-A)\mathbf{x} = \lambda D\mathbf{x}$ which translates into the condition that at each vertex v

$$d(v)\mathbf{x}(v) - \sum_{u\sim v} w(u,v)\mathbf{x}(u) = \lambda d(v)\mathbf{x}(v). \tag{2}$$

We will be showing that two graphs share common eigenvalues by showing that relationship (1) or (2) (depending on which matrix we are considering) holds for a given value of λ on both graphs.

2 2-edge-coverings and the adjacency matrix

In this section and the next we will illustrate the techniques of how to calculate the eigenvalues of a graph G which has a 2-edge-covering (i.e., there is some H so that $\pi : V(G)\to V(H)$ is a 2-edge-covering). We will use the graph in Figure 3 for an example of how to apply the techniques.

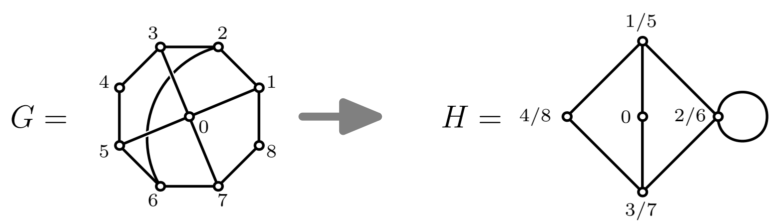

Figure 3: Our example 2-edge-covering. [All edge weights are 1.]

To find the eigenvalues of the adjacency matrix of G we will make use of two graphs. The first is a modified cover graph, denoted $H_\circ$ (to keep track of which graph we will be dealing with, anytime we are referring to $H_\circ$ we will use the $\Box_\circ$ notation). The graph will have the same vertices and the only modification will be to the weight function as follows:

$$w_\circ(u_\circ, v_\circ) = \begin{cases} \sqrt{2}w(u,v) & u \text{ or } v \text{ is a folding vertex;} \\ w(u,v) & \text{otherwise.} \end{cases}$$

The second graph will be an anti-cover graph and we denote it by H° (again anytime we refer to H° we use the $\square^\circ$ notation). The first step to defining the anti-cover is to give a sign function on the vertices of G so that for each v, $\operatorname{sgn}(v) \in \{-1, 0, 1\}$ where $\operatorname{sgn}(v) = 0$ if and only if the vertex folds, otherwise if $\pi(u) = \pi(v)$ for $u \neq v$ then $\operatorname{sgn}(u) = -\operatorname{sgn}(v)$.

Then an anti-cover H° is formed by removing all folding vertices and incident edges, for any remaining edge $u^\circ \sim v^\circ$ which is covered by edge $u \sim v$, the edge weight will be

$$w^\circ(u^\circ, v^\circ) = w(u, v) \operatorname{sgn}(u) \operatorname{sgn}(v). \tag{3}$$

Similar to Lemma 1 it can be shown that this weight function is well defined, i.e., choosing either of the two edges which cover an edge will give the same result. Also we note that signing of the graph, and so also the anti-cover, is not unique, but it can be shown that the adjacency matrix of two different anti-cover graphs are similar by a diagonal matrix with ± 1 on the diagonal.

In the special case that there are no vertices which fold then $H_\circ = H$ and H° is a signed version of H. This case has been previously considered by D'Amato [5] and more recently Bilu and Linial [1].

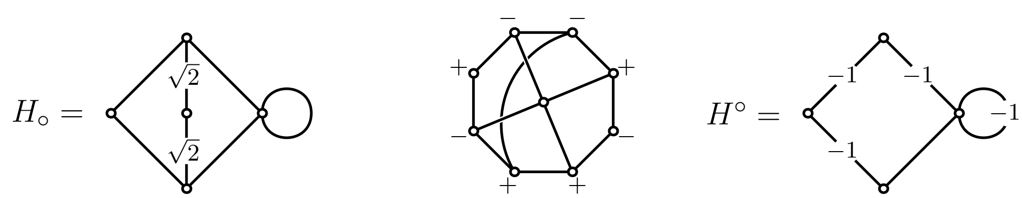

Figure 4: The graphs $H_\circ$, a signing of G, and corresponding H° for graphs in Figure 3. (Unmarked edges have weight 1.)

Theorem 2. *If G has a 2-edge-covering of H then the eigenvalues of the adjacency matrix of G is the union of the eigenvalues of the adjacency matrices of $H_\circ$ and H° (counting multiplicity).*

Proof. We will show how to lift eigenvectors of $H_\circ$ and H° to be eigenvectors of G. Then since the number of vertices (and hence) eigenvectors of $H_\circ$ and H° combined is equal to the number of vertices of G, and the independence of the vectors will follow from the construction the result will follow.

So suppose that λ is an eigenvalue of $H_\circ$ with eigenvector $\mathbf{x}_\circ$, then consider the following vector defined for G by

$$\mathbf{x}(u) = \begin{cases} \sqrt{2}\mathbf{x}_\circ(u_\circ) & u \text{ is a folding vertex;} \\ \mathbf{x}_\circ(u_\circ) & \text{otherwise;} \end{cases}$$

where $u_\circ = \pi(u)$.

We now show that (1) holds for $\mathbf{x}$ showing that λ is an eigenvector of G, we do this by considering two cases.

(1) v is not a folding vertex:

$$\begin{aligned}\sum_{u\sim v} w(u,v)\mathbf{x}(u) &= \sum_{\substack{u\sim v\\ u \text{ folds}}} w(u,v)\mathbf{x}(u) + \sum_{\substack{u\sim v\\ u \text{ not fold}}} w(u,v)\mathbf{x}(u)\\ &= \sum_{\substack{u_\circ\sim v_\circ\\ u_\circ \text{ folds}}} \frac{w_\circ(u_\circ,v_\circ)}{\sqrt{2}}\big(\sqrt{2}\mathbf{x}_\circ(u_\circ)\big) + \sum_{\substack{u_\circ\sim v_\circ\\ u_\circ \text{ not fold}}} w_\circ(u_\circ,v_\circ)\mathbf{x}_\circ(u_\circ)\\ &= \sum_{u_\circ\sim v_\circ} w_\circ(u_\circ,v_\circ)\mathbf{x}_\circ(u_\circ) \;=\; \lambda\mathbf{x}_\circ(v_\circ) \;=\; \lambda\mathbf{x}(v).\end{aligned}$$

(2) v is a folding vertex:

$$\sum_{u\sim v} w(u,v)\mathbf{x}(u) \;=\; 2\sum_{u_\circ\sim v_\circ}\frac{w_\circ(u_\circ,v_\circ)}{\sqrt{2}}\mathbf{x}_\circ(u_\circ) \;=\; \sqrt{2}\lambda\mathbf{x}_\circ(v_\circ) \;=\; \lambda\mathbf{x}(v).$$

Similarly, now suppose that λ is an eigenvalue of H° with eigenvector $\mathbf{x}^\circ$, then consider the following vector defined for G by

$$\mathbf{x}(u) = \begin{cases} 0 & u \text{ is a folding vertex;}\\ \operatorname{sgn}(u)\mathbf{x}^\circ(u^\circ) & \text{otherwise.}\end{cases}$$

We again show that (1) holds for $\mathbf{x}$ showing that λ is an eigenvector of G, by considering two cases.

(1) v is not a folding vertex:

$$\begin{aligned}\sum_{u\sim v} w(u,v)\mathbf{x}(u) &= \sum_{\substack{u\sim v\\ u \text{ not folds}}} w(u,v)\mathbf{x}(u) + \sum_{\substack{u\sim v\\ u \text{ folds}}} w(u,v)\mathbf{x}(u)\\ &= \operatorname{sgn}(v)\sum_{u^\circ\sim v^\circ} w^\circ(u^\circ,v^\circ)\mathbf{x}^\circ(u^\circ)\\ &= \lambda\operatorname{sgn}(v)\mathbf{x}^\circ(v^\circ) = \lambda\mathbf{x}(v). \qquad (4)\end{aligned}$$

(2) v is a folding vertex:

$$\begin{aligned}\sum_{u\sim v} w(u,v)\mathbf{x}(u) &= \sum_{\substack{u\sim v\\ \operatorname{sgn}(u)=1}} w(u,v)\mathbf{x}(u) + \sum_{\substack{u\sim v\\ \operatorname{sgn}(u)=-1}} w(u,v)\mathbf{x}(u)\\ &= \sum_{\substack{u\sim v\\ \operatorname{sgn}(u)=1}} w(u,v)\mathbf{x}^\circ(u^\circ) - \sum_{\substack{u\sim v\\ \operatorname{sgn}(u)=-1}} w(u,v)\mathbf{x}^\circ(u^\circ)\\ &= 0 \;=\; \lambda\mathbf{x}(v). \qquad (5)\end{aligned}$$

Thus the eigenvalues of $H_\circ$ and H° are eigenvalues of G and, as noted above, this concludes the proof. $\square$

3 2-edge-coverings and the normalized Laplacian

We again consider the problem of how to calculate the eigenvalues of G but this time for the normalized Laplacian. We will again make use of two graphs, but this time they will be slightly different. First, we will let $H_\Delta = H$.

The second graph H^Δ is again found by removing the folding vertices and incident vertices, and also taking a sign function on G as before and defining $w^\Delta(u^\Delta, v^\Delta)$ similarly to (3). There is one additional structure that we will need for H^Δ and that is a weight function on the vertices,

$$w^\Delta(v^\Delta) = \sum_{\substack{u\sim v \\ u \text{ folds}}} w(u,v).$$

This new weight function shows up in the degrees in H^Δ which are defined as follows,

$$d^\Delta(v^\Delta) = w^\Delta(v^\Delta) + \sum_{u^\Delta\sim v^\Delta} \left|w^\Delta(u^\Delta, v^\Delta)\right|.$$

Intuitively, $w^\Delta(v^\Delta)$ is used to correct for the change caused by the removal of edges incident to folding vertices so that now $d(v) = d^\Delta(v^\Delta)$. Pictorially, we will note a weight at vertex v^Δ of k by putting " $\boxed{k}$ " at the vertex.

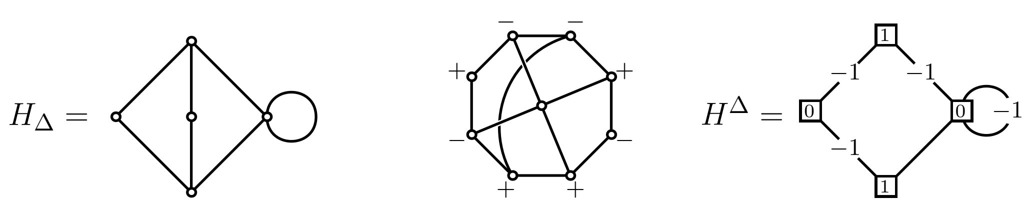

Figure 5: The graphs H_Δ, a signing of G, and corresponding H^Δ for graphs in Figure 3. (Unmarked edges have weight 1.)

Theorem 3. *If G has a 2-edge-covering of H then the eigenvalues of the normalized Laplacian matrix of G is the union of the eigenvalues of the normalized Laplacian matrices of $H = H_\Delta$ and H^Δ (counting multiplicity).*

The proof of Theorem 3 works the same as Theorem 2 and so will be omitted.

3.1 Can you hear an anti-cover graph?

A famous question in spectral graph theory is "can you hear the shape of a graph?" That is, given the eigenvalues can you determine the graph that produced them. There are many examples of two graphs that share the same spectrum but are not the same graph, while there are also examples of graphs that are uniquely determined by the spectrum.

But when it comes to the normalized Laplacian of the anti-covers H^{Δ} the situation can be even worse. Consider the two weighted graphs in Figure 6. These graphs

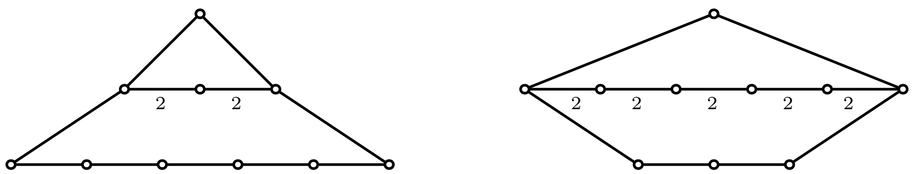

Figure 6: Two graphs with several shared eigenvalues. (Unmarked edges have weight 1.)

share four nontrivial eigenvalues. They also have obvious left/right symmetry and the eigenvalues found by the 2-edge-covering when the graphs are folded in half are the ones which are not common. Thus the shared eigenvalues come from their anti-cover graphs. These are simple to construct and are shown in Figure 7. However

Figure 7: Anti-covers of the graphs shown in Figure 6

it is not possible to obtain one of these anti-coverings from the other by relabeling and/or scaling. So the shared eigenvalues are the result of two cospectral anti-covers. However the situation is a little more interesting than that. If we now compute $\mathcal{L}^{\Delta}$ for both these graphs they both give

$$\begin{pmatrix} 1 & \frac{-1}{2\sqrt{2}} & 0 & 0 \\ \frac{-1}{2\sqrt{2}} & 1 & \frac{-1}{2} & 0 \\ 0 & \frac{-1}{2} & 1 & \frac{-1}{2} \\ 0 & 0 & \frac{-1}{2} & \frac{3}{2} \end{pmatrix}.$$

In particular, the anti-cover graph cannot in general be uniquely determined from the corresponding normalized Laplacian of the anti-cover. So these graphs are not only co-spectral they are also co-normalized Laplacian.

4 Conclusions

Returning to the graphs in Figure 1 it is now easy to find the (modified) cover and anti-cover graphs of all three graphs (where the 2-edge-covering is given by folding in half along the vertical axis). Moreover for both the adjacency matrix and the normalized Laplacian all three have the same anti-cover (the path of length four in the first case and the path of length four with all vertex weights 1 in the second) and thus they all share four eigenvalues in common.

In general, we have considered graphs G which are 2-edge-coverings of another graph and shown how to find the spectrum of the adjacency matrix and normalized Laplacian G by finding the corresponding spectrum of two smaller graphs.

For directed graphs it is easy to adapt the definition of a 2-edge-covering and the constructions and proof given in Section 2 to establish the following theorem. (We omit the definitions, but they are the obvious generalizations of what we have already given.)

Theorem 4. *If $\vec{G}$ has a 2-edge-covering of $\vec{H}$ then the eigenvalues of the adjacency matrix of $\vec{G}$ contains the union of the eigenvalues of the adjacency matrices of $\vec{H}_\circ$ and $\vec{H}^\circ$ (counting multiplicity of the respective eigenspaces).*

It would be interesting to determine if the eigenvalues of $\vec{G}$ was the union of the eigenvalues of $\vec{H}_\circ$ and $\vec{H}^\circ$. The difficulty lies in that for directed graphs the adjacency matrices do not need to have a full set of eigenvectors which are used in the proof. A similar result for directed graphs for the results in Section 3 is more problematic as there is often no well defined normalized Laplacian for a directed graph, see for example Chung [4] and Butler [2].

One can consider 3-edge-coverings and more generally k-edge-coverings. In this direction D'Amato [6] has considered a special type of 3-edge-covering. It would be nice to have a complete theory for an arbitrary k-covering to find all of the eigenvalues.

In Section 3 we saw that the graph $H^\triangle$ had both negative edge weights and a weight function on the vertices. While signed graphs and the combinatorial Laplacian have previously been investigated by Hou and Li [8] no investigation into these types of graphs for the normalized Laplacians has occurred. It would be interesting to see what properties of the normalized Laplacian of $H^\triangle$ still hold. For instance it is easy to show that all the eigenvalues of $H^\triangle$ still lie in the closed interval between 0 and 2.

References

[1] Y. Bilu and N. Linial, Lifts, discrepancy and nearly optimal spectral gap, *Combinatorica* **26** (2006), 495–519.

[2] S. Butler, Interlacing for weighted graphs using the normalized Laplacian, *Electronic Journal of Linear Algebra* **16** (2007), 90–98.

[3] F. Chung, Spectral Graph Theory, CBMS Lecture Notes, AMS, Providence, RI, 1997.

[4] F. Chung, Laplacians and the Cheeger inequality for directed graphs, *Annals of Combinatorics* **9** (2005), 1–19.

[5] S. S. D'Amato, Eigenvalues of graphs with twofold symmetry, *Molecular Physics* **37** (1979), 1363–1369.

[6] S. S. D'Amato, Eigenvalues of graphs with threefold symmetry, *Theoretica Chimica Acta* **53** (1979), 319–326.

[7] C. Godsil and G. Royle, Algebraic Graph Theory, Springer-Verlag, New York, 2001.

[8] Y. P. Hou and J. S. Li, On the Laplacian eigenvalues of signed graphs, *Linear and Multilinear Algebra* **51** (2003), 21–30.