Lecture 1 – March 30

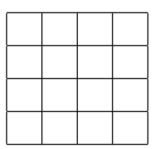
We start our investigation into combinatorics by focusing on *enumerative* combinatorics. This topic deals with the problem of answering "how many?" We will start with a set S, which is a collection of objects called elements, and then we will determine the number of elements in our set denoted $|S|$. (More information about the basics of set theory are found in the appendix of *Applied Combinatorics*.)

We start with a few basic rules for counting.

<u>Addition Rule</u>
If the set we are counting can be broken into *disjoint* pieces, then the size of the set it the *sum* of the size of the pieces.

$$S = \bigcup_{i=1}^{m} S_i,\ S_i \cap S_j = \emptyset \text{ if } i \neq j \ \Rightarrow\ |S| = \sum_{i=1}^{m} |S_i|.$$

Example: A square with side length 4 is divided into 16 equal sqaures as shown below. What is the total number of squares in the picture?



Solution: We break the squares up according to size. There are sixteen 1×1 squares, nine $2\times w$ squares, four 3×3 squares and one 4×4 square. So by the addition rule there is a total of $16+9+4+1=30$ squares. (Note: here we must be careful about what is meant by disjoint. Geometrically as squares the squares are not all disjoint one from another, but we are not concerned with the geometry of the problem so when we talk about disjoint we mean not the same square. So in our case when we broke up the squares into these sets by size theses sets are disjoint one from another.)

<u>Multiplication Rule</u>
Suppose we can describe the elements in our set using a procedure with m steps, where at the ith step we have r_i choices available. (Where the *number* of choices is independent of our previous choices.) Then the size of the set is $r_1 r_2 \cdots r_m$.

It is important to realize that what we are counting using the multiplication rule is the number of choices that can be made. In order to guarantee that we get the right count it is important that (1) every element in our set is realizable by at least one set of choices and (2) no two sets of choices gives us the same element. An example of what can go wrong will be given in a later lecture.

Another important thing to note is that the number of choices is independent on the previous choices but not necessarily the choice that we have to make. This is illustrated in the next example.

Example: How many standard California license plates are possible? A standard license plate has the form $NLLLNNN$ where N is a number and L is a letter. How many are possible if there is no repetition of numbers or letters?

Solution: We can form the license plate one character at a time. So we will use the multiplication rule where the decision at the ith stage is what character goes in the ith slot. Since there are 10 possible numbers and 26 possible letters we have that the number of license plates is

$$10\cdot26\cdot26\cdot26\cdot10\cdot10\cdot10 = 175,760,000.$$

When we add the restriction that that there is no repetition of characters we again use the same principle as before. Now the difference is that when we fill in the second number we only have nine options since we have already used one number, when we fill in the third we only have eight options and when we fill in the fourth we only have seven. Note that here we do not know which nine, eight or seven choices are available because that depends on previous choices, but the number of choices does not! So the number when there is no repetition of numbers is

$$10\cdot26\cdot25\cdot24\cdot9\cdot8\cdot7 = 78,624,000.$$

<u>Bijection Rule</u>
If the elements in S can be paired in a one-to-one (i.e., bijective) fashion, then $|S| = |T|$.

We have already used the bijection rule when we did the multiplication rule, since what we are doing is counting the number of choices which are paired one-to-one with the elements. One interesting thing is that it is possible to show that two sets have equal size by giving a bijection between the sets without ever knowing the size of the sets themselves.

Example: How many subsets of $[n] = \{1, 2, \dots, n\}$ are there?

Solution: We can pair a subset of $[n]$ with a binary word of length n (a binary word is a word with the individual letters coming from 0 and 1). The way this is done is by taking a subset A and letting the ith letter of the binary word be 1 if i is in A and 0 if i is not in A. For example for $\{1, 2, 3\}$ we have

$\emptyset \leftrightarrow 000$	$\{3\} \leftrightarrow 001$
$\{1\} \leftrightarrow 100$	$\{1,3\} \leftrightarrow 101$
$\{2\} \leftrightarrow 010$	$\{2,3\} \leftrightarrow 011$
$\{1,2\} \leftrightarrow 110$	$\{1,2,3\} \leftrightarrow 111$

Since there are 2^n binary words it follows by the bijection rule that the number of subsets is also 2^n.

> Rule of Counting in Two Ways
>
> If we count S in two different ways the results must be equal.

This is the basic idea behind many combinatorial proofs. Namely we find an appropriate set and count it in two different ways. By then setting them equal we get some nontrivial relationships.

Example: Count the number of $*$'s in the diagram which has n rows and $n+1$ columns (below we show the case $n = 4$) in two different ways.

$$\begin{matrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{matrix}$$

Solution: For our first method we count along rows. Each row has $(n+1)$ of the $*$'s and there are n rows so in total we have $n(n+1)$ total $*$'s. For our second method we count using the diagonals. In particular we have that there are

$$1+2+3+\cdots+n+n+\cdots+3+2+1 = 2\sum_{i=1}^{n} i.$$

Equating these two ways of counting we have

$$n(n+1) = 2\sum_{i=1}^{n} i \quad \text{or} \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

> Probability is counting
>
> Probability is a measurement of how likely an outcome is to occur. If we are dealing with finitely many possible outcomes and each outcome is equally likely (very common assumptions!) then the probability that a certain outcome occurs is the ratio of the outcomes with the desired result divided by the number of all possible outcomes.

Lecture 2 – April 1

Example: A domino is a tile of size 1×2 divided into two squares. On each square are pips (or dots), the number of pips possible are$0, 1, 2, 3, 4, 5, 6$. There are 28 possible domino pieces, i.e., $\boxed{0 \mid 0}$, $\boxed{0 \mid 1}$, $\boxed{0 \mid 2}$, $\ldots$, $\boxed{6 \mid 6}$. Two domino pieces fit if we can arrange them so the two adjacent squares have the same number of pips. So for example $\boxed{1 \mid 3}\,\boxed{3 \mid 6}$ fit but $\boxed{1 \mid 3}\,\boxed{2 \mid 6}$ do not fit. What is the probability that two dominos chosen at random will fit?

Solution: First let us count the total number of ways to pick two dominos. We can think of choosing one domino and then a second one. This can be done in $28{\cdot}27 = 756$ ways. But we have overcounted, this is because the order that we pick domino pieces should not matter. So for example right now we distinguish between the choice $\boxed{1 \mid 3}\,\boxed{5 \mid 6}$ and the choice $\boxed{5 \mid 6}\,\boxed{1 \mid 3}$. In essence we have counted every pair twice. To correct for this we divide by 2 so that the number of ways to pick two domino pieces is $756/2 = 378$.

Next let us count the number of pairs for which the dominos fit. We again think of picking a domino and then seeing how many domino pieces fit with it. There are two cases, if we first pick a domino of the form $\boxed{x \mid x}$ (of which there are seven of this form) then dominos which fit with it are of the form $\boxed{x \mid y}$ with $y \neq x$, so there are six matches. So in this case we got $6{\cdot}7 = 42$ pairs. In the second case we have a domino of the form $\boxed{x \mid y}$ where $x \neq y$, there are 21 such dominos and there are twelve dominos that can fit with this type of domino, namely, $\boxed{x \mid z}$ for $z \neq x$ and $\boxed{y \mid z}$ for $z \neq y$. So in this case we got $21{\cdot}12 = 252$ pairs. Combining we have $42 + 252 = 294$ pairs that match, but as before we have counted every pair twice and so there are 147 pairs of dominos that fit.

Combining we have that the probability that two dominos chosen at random will fit is $147/378 = 7/18$.

Given a set with n objects, a *permutation* is an ordered arrangement of all n of the objects. An *r-permutation* is an ordered arrangement of r of the n objects. An *r-combination* is an unordered selection of r of the n objects.

We now count how many of each type of the following there are. First off for a permutation by the multiplication rule we choose which object will be first (we have n choices), then we choose which object will be second (we now have $n-1$ choices), and continue so on to the end. So there are

$$n(n-1)(n-2)\cdots 3{\cdot}2{\cdot}1 = n! \quad \text{permutations.}$$

The notation $n!$, read "n factorial", arises frequently in combinatorics and it is important to get a good handle on using it. For example, it is useful to be able to rewrite factorials, i.e., $(2n+2)! = (2n+2)(2n+1)(2n)!$. For convenience we will define $0! = 1$ (this helps to simplify a lot of notation).

Another useful fact for factorials is Stirling's approximation which states that

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

This is useful when trying to get a handle on the size of an expression that involves factorial. For instance, one object which has been extensively studied are the middle binomial coefficients (see below) $\binom{2n}{n}$, which by Sterling's approximation we have

$$\binom{2n}{n} = \frac{(2n)!}{n!n!} \approx \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}} = \frac{1}{\sqrt{\pi n}}4^n.$$

The method to count r-permutations is the same as that of counting permutations. We use the multiplication principle but instead of going through n stages we stop at the rth stage. So we have that there are

$$\begin{aligned} &n(n-1)(n-2)\cdots\big(n-(r-1)\big) \\ &= \frac{n(n-1)(n-2)\cdots\big(n-(r-1)\big)(n-r)\cdots 2\cdot 1}{(n-r)\cdots 2\cdot 1} \\ &= \frac{n!}{(n-r)!} = P(n,r). \end{aligned}$$

This is also sometimes written as the falling factorial $(n)_k$ or $n^{\underline{k}}$.

To count r-combinations we can count r-permutations in a different way. Suppose that we let $C(n,r)$ be the number of ways to pick r unordered objects, then we have

$$P(n,r) = C(n,r)r!.$$

Namely, to have an ordering of r elements we first pick the r elements and then we order them. Rearranging we have

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}.$$

The term $\binom{n}{r}$, read "n choose r", is also known as a binomial coefficient. This will be discussed in a future lecture.

Example: How many ways can n rooks be placed on $n \times n$ chessboard so no two threaten each other? How many ways can k rooks be placed on $n \times n$ chessboard if $k \le n$?

Solution: A rook can move along any row and column. If there are n rooks and at most one rook can go in a column then there will be a single rook in each column. We place rooks one column at a time. In the first column we have n choices, in the second column we cannot put it in the same row as we used for the first rook so there are $n-1$ choices, for the third column we have $n-2$ choices and so on. In total there are

$$n(n-1)(n-2)\cdots 2\cdot 1 = n! \quad \text{placements}.$$

If we have k rooks then the approach is similar. First we choose the columns that the rooks will go into. This can be done in $C(n,k)$ ways. Once we have the columns we place rooks in one column at a time. In particular the way that we can distribute the rooks in the k columns is

$$n(n-1)\cdots\big(n-(k-1)\big) = P(n,k).$$

Putting it together the number of ways to place the k rooks is $C(n,k)P(n,k)$.

Problems of placing rooks on chessboards can lead to some interesting (and highly non-trivial) combinatorics! These types of problems can arise in algebraic combinatorics which unfortunately we will not have time to address this quarter.

Example: Given eight points which line in the plane, no three on a line, how many lines do these points determine?

Solution: To determine a line we need two points. So this reduces to the number of ways to pick two of the eight patterns which can be done in $\binom{8}{2}$ ways. Note that it is important that no three are on a line to guarantee that we do not overcount!

Lecture 3 – April 3

Example: How many ways are there to choose k elements from $[n] = \{1, 2, \ldots, n\}$ so that no *two* are consecutive?

Solution: If we did not have the restriction that elements couldn't be consecutive then we could do this in $\binom{n}{k}$ ways. The problem is on how to guarantee that the elements are not consecutive. Suppose that our elements we choose are

$$a_1 < a_2 < a_3 < \cdots < a_k$$

where no two are consecutive. Using the fact that no two are consecutive we have that

$$a_1 < a_2 - 1 < a_3 - 2 < \cdots < a_k - (k-1).$$

These are k ditinct elements from $[n-(k-1)] = \{1, 2, \ldots, n-(k-1)\}$. On the other hand given k distinct elements $b_1 < b_2 < \cdots < b_k$ from $[n-(k-1)]$ we can find k elements from $[n]$ for which no two are consecutive, namely the elements

$$b_1 < b_2 + 1 < b_3 + 2 < \cdots < b_k + (k-1).$$

So the number of ways there are to choose k elements from $[n] = \{1, 2, \ldots, n\}$ so that no *two* are consecutive is the same as the number of ways there are to choose k elements from $[n-(k-1)]$, which is $\binom{n-(k-1)}{k}$.

When using the multiplication principle it is important to make sure that different sets of choices do not lead to the same outcome. This can sometimes be subtle to detect so great care should be taken!

Example: From among seven boys and four girls how many ways are there to choose a six member volleyball team with at least two girls?

Solution: One (seemingly) natural approach is to first pick two girls and then pick the remainder of the team. This way we guarantee that we meet the condition. The number of ways to pick the two girls is $\binom{4}{2}$, there are now nine people left and we have to choose 4 of then and this could be done in $\binom{9}{4}$ ways, for a grand total of $\binom{4}{2}\binom{9}{4} = 756$ different possible teams.

But wait! Suppose the girls are A, B, C, D and the boys $1, 2, 3, 4, 5, 6, 7$ then we could have first chosen AB and then $C246$ to form the team $ABC246$, but we could also have first chosen BC and then $A246$ to form the team $ABC246$. So different choices gave us the same team. (We have committed the cardinal sin of overcounting.)

To correct this we have two options, one is to figure out how much we have overcounted and then subtract off the overcount. The other is to go back and count in a different way, one such different method is to split up the count into smaller cases where we won't overcount. For example in this problem let us break it up by the number of girls on the team. If there are two girls we pick two girls and four boys to form the team which can be done in $\binom{4}{2}\binom{7}{4}$, if there are three girls we pick three girls and three boys to form the team which can be done in $\binom{4}{3}\binom{7}{3}$, if there are four girls we pick four girls and two boys to form the team which can be done in $\binom{4}{4}\binom{7}{2}$ ways, so altogether there are

$$\binom{4}{2}\binom{7}{4} + \binom{4}{3}\binom{7}{3} + \binom{4}{4}\binom{7}{2} = 371 \quad \text{teams.}$$

An arrangement of n distinct objects is $n!$. But what happens if the objects are not distinct? For example, suppose we are looking at anagrams, rearrangements of letters of a word. For instance "STEVE BUTLER" can be arranged to "RUE VEST BELT" but it could also be arranged to gibberish "VTSBEEERTLU". So the number of ways there are to rearrange the letters of a given word (allowing for gibberish answers) is found by counting the arrangements of objects with repetition.

Example: How many ways are there to rearrange the letters of the word "BANANA"?

Solution: We present two methods. The first method is to group the letters together by type. We have one B, three As and two Ns. We now start with six empty slots which we will fill in with our letters in order. First we put in the B, there are six slots and we must choose one of them so this can be done in $\binom{6}{1}$ ways. We now have five slots left for us to choose for the position of the three As which can be done in $\binom{5}{3}$ ways. Finally we have two slots left for us to choose for the position of the two Ns which can be done in $\binom{2}{2}$ ways. Giving us a total of

$$\binom{6}{1}\binom{5}{3}\binom{2}{2} = 60 \quad \text{rearrangements.}$$

For our second method we first make the letters distinct. This is done by labeling, so we have the letters $B, A_1, A_2, A_3, N_1, N_2$. These letters can be arranged in $6!$ ways and finally we remove the labaling. The problem is that we have now overcounted. For instance for the end result of $ABANNA$ this would show up twelve ways, as

$$\begin{array}{lll} A_1BA_2N_1N_2A_3 & A_1BA_3N_1N_2A_2 & A_2BA_1N_1N_2A_3 \\ A_2BA_3N_1N_2A_1 & A_3BA_1N_1N_2A_2 & A_3BA_2N_1N_2A_1 \\ A_1BA_2N_2N_1A_3 & A_1BA_3N_2N_1A_2 & A_2BA_1N_2N_1A_3 \\ A_2BA_3N_2N_1A_1 & A_3BA_1N_2N_1A_2 & A_3BA_2N_2N_1A_1 \end{array}$$

Namely, we have 3! ways to label the As and 2! ways to label the Ns and 1! ways to label the B. This happens with each arrangement so in total we have

$$\frac{6!}{1!3!2!} = 60 \quad \text{rearrangements.}$$

These two approaches generalize. The first idea is to choose the slots for the first type of objects, then choose the slots for the second type of objects and so on. The second idea is to label the elements and permute and then divide out by the overcounting.

Given n_1 objects of type 1, n_2 objects of type 2, ..., n_k objects of type k and $n = n_1 + n_2 + \cdots + n_k$ then the number of ways to arrange these objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} = \frac{n!}{n_1!n_2!\cdots n_k!} = \binom{n}{n_1, n_2, \ldots, n_k}.$$

The term $\binom{n}{n_1, n_2, \ldots, n_k}$ is also known as a multinomial coefficient, we will discuss these more in a later lecture.

Example: How many ways are there to rearrange the letters of "ROKOKO" so there in no "OOO"?

Solution: The number of ways to rearrange the letters of "ROKOKO" is the same as the number of ways to rearrange the letters of "BANANA", so there are 60 in all. But of course some of these have "OOO" but we can use one of the most useful techniques in counting:

Sometimes it is easier to count the complement.

So we now count the number of arrangements with "OOO". This can be done by considering the arrangements of the letters R,K,K,OOO. This can be done in $4!/2! = 12$ ways. So the total number of rearrangements without "OOO" is $60 - 12 = 48$.

A related problem is to distribute n identical objects among k different boxes (or people, or days, or so on). One way to do this is to use $*$'s (stars) to denote the identical objects and lay them down in a row. We then draw in $k-1$ dividing lines (bars) which divides the n objects into k groups, the first group goes in the first box, the second group goes in the second box and so on.

For example suppose we are distributing $n = 20$ pennies among $k = 6$ children. Then using bars and stars we represent giving four pennies to the first child, two to the second, six to the third, none to the fourth, five to the fifth and three to the sixth by

$$* * * * \mid * * \mid * * * * * * \mid \mid * * * * * \mid * * * .$$

In this case we have a total of 25 bars and stars and once we know where the bars (or stars) go then we know where the stars (or bars) go. So the number of ways to do the distribution is the number of ways to pick the bars (or stars) which is $\binom{25}{5} = \binom{25}{20}$.

> The number of ways to distribute n identical objects into k distinguished boxes is
>
> $$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}.$$

Example: How many ways are there to distribute twenty pennies among six children? What if each child must get at least one penny?

Solution: We already have answered the first part, this can be done in $\binom{25}{5} = 53,130$ ways. For the second part we distribute pennies in two rounds. In the first round we give one penny to each child, thus satisfying the condition that each child gets a penny. In the second round we distribute the remaining fourteen pennies among the six children arbitrarily which can be done in $\binom{19}{5} = 11,628$ ways.

Lecture 4 – April 6

Example: How many ways are there to order six 0s, five 1s and four 2s so that the first 0 occurs before the first 1 which in turn occurs before the first 2?

Solution: We can build up the words one group of letters at a time. To simplify the process we first put down the 2s, then the 1s and finally the 0s. First putting down the 2s we have

2222

Next we put down the 1s which can go before and after the 2s, i.e.,

␣2␣2␣2␣2␣.

To satisfy the constraint we must have one of the 1s go into the first slot, and the remaining $n = 4$ 1s are distributed among the $k = 5$ slots which can be done in $\binom{8}{4}$ ways. We now need to decide how to put in the 0s, these again can go before and after the letters already placed, i.e.,

␣1␣*␣*␣*␣*␣*␣*␣*␣*␣.

To satisfy the constraint we must have one of the 0s go into the first slot, and the remaining $n = 5$ 0s are distributed among the $k = 10$ slots which can be done in $\binom{14}{9}$ ways. Combining this gives us

$$\binom{8}{4}\binom{14}{9} = 140,140 \quad \text{ways.}$$

Many problems into combinatorics relate to the problem of placing n balls into k bins. The number of ways to do this is dependent on our assumptions.

- *Identical balls and distinct bins:* This is what was discussed in the previous lecture, so it can be done in
$$\binom{n+(k-1)}{k-1} \quad \text{ways.}$$
- *Distinct balls into distinct bins:* Imagine we place the balls one at a time. The first ball can go into any of k bins, the second ball can go into any of k bins, ..., the nth ball can go into any of k bins. So by the multiplication rule the number of ways that this can be done is
$$\underbrace{k \cdot k \cdot \cdots \cdot k}_{n \text{ times}} = k^n.$$
- *Distinct balls into distinct bins, with number of balls in each bin specified:* That is we want to place the balls so that n_1 balls go into the first bin, n_2 balls go into the second bin, ..., n_k balls go into the kth bin, where $n = n_1 + \cdots + n_k$. This can be done by choosing n_1 balls for the first bin, then choose n_2 of the remaining balls for the second bin, n_3 of the now remaining balls for the third bin, and so on. The number of ways to do this is
$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k},$$
which from the last lecture is the same as
$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

- *Identical balls into identical bins:* This looks at how to break up the n balls into groups where order is unimportant. This leads to the field of *partitions* (an area of combinatorics with really beautiful proofs and really hard problems), we will be looking at these in a later lecture.

- *Distinct balls into identical bins:* This looks at how to break up $[n] = \{1, 2, \ldots, n\}$ into small subsets. This can be counted using Bell numbers and Stirling numbers which we will look at (briefly) in a later lecture.

The number of ways of distributing n identical objects into k bins has another interpretations, namely the number of nonnegative integer solutions to

$$x_1 + x_2 + \cdots + x_k = n.$$

To see this we count the balls, let x_i be the number of balls in the ith bin, then the sum of the x_i is the total number of balls which is n. Conversely, given a solution of x_i we can make a distribution into the bins by putting x_i balls into the ith bin for each i.

Example: Count the number of integer solutions for each of the following.

1. $x_1 + x_2 + x_3 + x_4 = 10$ with $x_i \geq 0$.
2. $x_1 + x_2 + x_3 + x_4 = 10$ with $x_i \geq 1$.
3. $x_1 + x_2 + x_3 + 3x_4 = 10$ with $x_i \geq 0$.
4. $x_1 + x_2 + x_3 + x_4 \leq 10$ with $x_i \geq 0$.

Solution: We have the following.

1. This is the straightforward case of $n = 10$ and $k = 4$ variables so this can be done in $\binom{13}{3} = 286$ ways.

2. We satisfy the constraint $x_i \geq 1$ by first distributing 1 to each x_i, we now distribute the remaining $n = 6$ among the $k = 4$ variables so this can be done in $\binom{9}{3} = 84$ ways.

3. The difficult part about this is the x_4 term, so we break into cases depending on the value of x_4. If $x_4 = 0$ we distribute $n = 10$ among $k = 3$ which can be done in $\binom{12}{2}$ ways. If $x_4 = 1$ we distribute $n = 7$ among $k = 3$ which can be done in $\binom{9}{2}$ ways. If $x_4 = 2$ we distribute $n = 4$ among $k = 3$ which can be done in $\binom{6}{2}$ ways. Finally, if $x_4 = 3$ we distribute $n = 1$ among $k = 3$ which can be done in $\binom{3}{2}$ ways. So by the addition rule the total number of solutions is

$$\binom{12}{2} + \binom{9}{2} + \binom{6}{2} + \binom{3}{2} = 120 \text{ ways.}$$

4. We could break this up as we did the previous case and get the sum of several terms. However we can avoid all of this by considering the auxiliary problem

$$x_1 + x_2 + x_3 + x_4 + x_5 = 10 \text{ with } x_i \geq 0.$$

Here the x_5 represents what we are "short" by in our sum to 10. In particular each solution to this problem is a solution to our original problem, giving us $\binom{14}{4} = 1,001$ ways.

Example: How many ways are there to rearrange the letters of the word "MATHEMATICASTER" so there are no consecutive vowels?

Solution: We break this into three steps. First the constraint gives us relationships between the consonants and the vowels so we will see how many vowel-consonant patterns there are. There are six vowels and nine consonants. If we put down the six vowels then we need to decide how to distribute the consonants among the vowels, i.e.,

␣V␣V␣V␣V␣V␣V␣.

To satisfy our constraint we have to put five of the consonants in slots between the vowels, we can distribute the remaining $n = 4$ consonants among the $k = 7$ slots which can be done in $\binom{10}{6}$ ways. Given the pattern we now arrange the vowels and consonants which can be done respectively in

$$\frac{6!}{3!2!1!} \text{ and } \frac{9!}{3!2!1!1!1!1!} \text{ ways.}$$

Combining everything we have

$$\binom{10}{6} \cdot \frac{6!}{3!2!1!} \cdot \frac{9!}{3!2!1!1!1!1!} = 381,024,000 \text{ ways.}$$

The *binomial theorem* looks at expressions of the form $(x + y)^n$. The term binomial refers to the two nomials, or variables, x and y. For the first few cases we have:

$$\begin{aligned}
(x+y)^0 &= 1 \\
(x+y)^1 &= x + y \\
(x+y)^2 &= x^2 + 2xy + y^2 \\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4
\end{aligned}$$

In general we have

$$\begin{aligned}
(x+y)^n &= \underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ terms}} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.
\end{aligned}$$

The idea behind the term $\binom{n}{k}$ is that the way we multiply out $(x+y)^n$ is to choose one term from each of the n copies of $(x+y)$, to get $x^k y^{n-k}$ we have to choose the x value for exactly k of the n possible places which can be done in $\binom{n}{k}$. Because of its connection to the binomial theorem the terms $\binom{n}{k}$ are also known as *binomial coefficients.*

There is a more general version of the binomial attributed to Isaac Newton. First we define for any number r (even imaginary!) and integer $k \geq 1$

$$\binom{r}{k} = \frac{r(r-1)\cdots\big(r-(k-1\big)}{k!}.$$

Further we let $\binom{r}{0} = 1$. Then for $|x| < 1$

$$(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k.$$

In the case that r is an integer then the term $\binom{r}{k}$ becomes 0 for k sufficiently large so this gives us back the form of the binomial theorem given above.

Lecture 5 – April 8

As an application of the binomial theorem we have the following.

> Let $e = 2.718281828\ldots$ and $0 \leq k \leq n$, then
> $$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

Proof: For the first inequality we have

$$\begin{aligned}\binom{n}{k} &= \frac{n}{k}\frac{(n-1)}{(k-1)}\frac{(n-2)}{(k-2)}\cdots\frac{n-(k-1)}{k-(k-1)} \\ &\geq \frac{n}{k}\frac{n}{k}\frac{n}{k}\cdots\frac{n}{k} \\ &= \left(\frac{n}{k}\right)^k,\end{aligned}$$

where we used the fact that $(n-a)/(k-a) \geq n/k$ for $0 \leq a \leq k-1$. For the other inequality we first note that from calculus we have that $e^x \geq 1+x$ for all x, In particular for $x \geq 0$ we have

$$\begin{aligned}e^{nx} &\geq (1+x)^n \\ &= \sum_{i=0}^{n} \binom{n}{i} x^i \\ &\geq \binom{n}{k} x^k.\end{aligned}$$

Now choose $x = k/n$ and simplify to get the result. □

Today we will look at using the binomial theorem and more generally at the binomial coefficients. Our starting point will be looking at the coefficients in the binomial theorem, i.e., $\binom{n}{k}$. These can be arranged in a triangle known as Pascal's triangle as follows:

$$\begin{array}{c} \binom{0}{0} \\ \binom{1}{0}\ \binom{1}{1} \\ \binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\ \binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\ \binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \\ \binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \\ \binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6} \end{array}$$

Inserting the values of these numbers we have

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \end{array}$$

These numbers occur frequently in combinatorics so it is good to have the first few rows memorized and know some of the basic properties of numbers in this triangle. We will now start to describe some of the various properties of the numbers in this triangle.

One thing which is apparent is that the rows are symmetric. In terms of the binomial coefficients this says the following.

> $$\binom{n}{k} = \binom{n}{n-k}.$$

Algebraic proof: Using the definition of $\binom{n}{k}$ from a previous lecture we have

$$\begin{aligned}\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{n!}{(n-k)!\big(n-(n-k)\big)!} = \binom{n}{n-k}.\end{aligned} \quad \square$$

Combinatorial proof: By definition $\binom{n}{k}$ is the number of ways to choose k elements from a set of n elements. This is the same as deciding which $n-k$ numbers will not be chosen which can be done in $\binom{n}{n-k}$ ways. □

We have now given two proofs. Certainly one proof is sufficient to show that it is true. Nevertheless it is useful to have many different proofs of the same fact (for instance there are hundreds of proofs for the Pythagorean Theorem and new ones are constantly being discvoered), since they can give us different ideas of how to use these tools in other problems. By "combinatorial proof" we mean a proof wherein we count some object in two different ways (i.e., using the *Rule of Counting in Two Ways* from the first lecture).

Another pattern that seems to be happening is that the terms in the rows first increase until the halfway point and then they decrease. This behavior is called unimodal and we have the following.

> For n fixed, $\displaystyle\binom{n}{k}$ is unimodal.

Proof: To show this we have to show that it first will increase and then decrease. This can be done by looking at the ratio of consecutive terms. In particular we have

$$\frac{\binom{n}{k}}{\binom{n}{k-1}}\begin{cases}\geq 1 & \text{then } \binom{n}{k}\geq\binom{n}{k-1},\\ \leq 1 & \text{then } \binom{n}{k}\leq\binom{n}{k-1}.\end{cases}$$

Substituting in the definition for the binomial coefficient we have

$$\frac{\binom{n}{k}}{\binom{n}{k-1}}=\frac{\frac{n!}{k!(n-k)!}}{\frac{n!}{(k-1)!(n-k+1)!}}=\frac{n-k+1}{k}.$$

Solving we see that $\binom{n}{k}/\binom{n}{k-1}\geq 1$ when $k\leq(n+1)/2$. In particular we see that for the first half of a row on Pascal's triangle that the binomial coefficients will increase and for the second half of the row they will decrease. □

Actually more can be said about the size of the binomial coefficients. Namely, it can be shown that they form the (infamous) bell shaped curve.

One very important pattern is how the coefficients of one row relate to the coefficients of the previous row. This gives us the most important identity for binomial coefficients.

$$\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}.$$

In other words this says to find the term $\binom{n}{k}$ in Pascal's triangle you look at the two numbers above and add them. Using this one can quickly generate the first few rows (so instead of memorizing the rows you can also memorize the first three or four and then memorize how to fill in the rest). This also can be used to give an inductive proof for the binomial theorem.

Algebraic proof: Plugging in the definitions for binomial coefficients we have

$$\begin{aligned}\binom{n-1}{k-1}+\binom{n-1}{k} &=\frac{(n-1)!}{(k-1)!(n-k)!}+\frac{(n-1)!}{k!(n-k-1)!}\\ &=\frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{1}{n-k}+\frac{1}{k}\right)\\ &=\frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{n}{k(n-k)}\right)\\ &=\frac{n!}{k!(n-k)!}=\binom{n}{k}.\end{aligned}$$ □

Combinatorial proof: We have that $\binom{n}{k}$ is the number of ways to select k elements from $\{1,2,\ldots,n\}$. Using the addition rule this is the number of ways to select k elements from $\{1,2,\ldots,n\}$ with n being one of the chosen elements added to the number of ways to select k elements from $\{1,2,\ldots,n\}$ with n not being one of the chosen elements. In the first case there are $\binom{n-1}{k-1}$ ways to choose the remaining $k-1$ elements other than n and in the second case there are $\binom{n-1}{k}$ ways to choose k elements other than n. □

If we sum the values in the first few rows of Pascal's triangle we see that we get $1,2,4,8,16,32,64$. This is a nice pattern and it holds in general.

$$\sum_{k=0}^{n}\binom{n}{k}=\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}=2^n.$$

Algebraic proof: Putting $x=y=1$ into the binomial theorem we have

$$2^n=(1+1)^n=\sum_{k=0}^{n}1^k1^{n-k}\binom{n}{k}=\sum_{k=0}^{n}\binom{n}{k}.$$ □

Combinatorial proof: In a previous lecture we saw that the number of subsets of $\{1,2,\ldots,n\}$ is 2^n. On the other hand the number of subsets with k elements is $\binom{n}{k}$. Combining these two ideas we have that the number of subsets is the number of subsets of size 0 added to the number of subsets of size 1 added to the number of subsets of size 2 … added to the number of subsets of size n. □

We can also use the binomial theorem in more subtle ways.

$$\sum_{k=0}^{n}k\binom{n}{k}=1\binom{n}{1}+2\binom{n}{2}\cdots+\binom{n}{n}=n2^{n-1}.$$

Algebraic proof: Thinking of the binomial theorem in terms of a function of x we have

$$n(x+1)^{n-1} = \frac{d}{dx}\big((x+1)^n\big) = \frac{d}{dx}\left(\sum_{k=0}^{n} x^k\binom{n}{k}\right) = \sum_{k=0}^{n} kx^{k-1}\binom{n}{k}.$$

Substituting $x = 1$ on the left and right sides gives us the result. □

Of course this also has a combinatorial interpretation.

Combinatorial proof: Examining the term $k\binom{n}{k}$ this counts the number of ways to pick a k-element subset and then pick one of these elements. We can interpret this as the number of ways of forming a committee of size k with a chairperson, i.e., first we pick the k people who will be on the committee and then we select one of the chosen people to be the chair. So the left hand side, which sums over all possible committee sizes, counts all possible ways to form a committee with a chairperson.

We can also count this by first selecting the chairperson (which can be done in n ways) and then choosing the rest of the committee. Since there are $n-1$ people available to serve on the committee with the chair we can fill in the rest of the committee in 2^{n-1} ways. □

Counting committees arrangements gives some simple arguments to some binomial identities. As another example of this type of committee forming argument consider the following.

$$\boxed{\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}.}$$

Combinatorial proof: Let us count the number of ways to form a committee of size r from a group of m women and n men. If we ignore the gender balance, since there are $m+n$ people then this can be done in $\binom{m+n}{r}$ ways. On the other hand the number of ways to form a committee with exactly k women is to first select the women in $\binom{m}{k}$ ways and then fill up the remaining seats with the men in $\binom{n}{r-k}$ ways. So $\binom{m}{k}\binom{n}{r-k}$ is the number of ways to form our committee of total size r with exactly k women. Since the possible values for k are $0, 1, 2, \ldots, r$ the result now follows by the rule of addition. □

There is also an algebraic proof. We do not give all the details but only provide a brief sketch: The right hand side is the coefficient of x^r in the expansion of $(1+x)^{m+n}$, the left hand side is the coefficient of x^r in the expansion of $(1+x)^m(1+x)^n$. Since $(1+x)^{m+n} = (1+x)^m(1+x)^n$ the coefficients must be equal giving the result.

As a special case of the previous result we have the following.

$$\boxed{\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.}$$

In other words if we add up the square of the numbers in a row of Pascal's triangle the result is another binomial coefficient. To see how this follows from what we just proved we note that using the symmetry of the binomial coefficients

$$\begin{aligned}
&\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 \\
&= \binom{n}{0}\binom{n}{0} + \binom{n}{1}\binom{n}{1} + \cdots + \binom{n}{n}\binom{n}{n} \\
&= \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n}\binom{n}{0} \\
&= \binom{2n}{n}.
\end{aligned}$$

The last step follows from noting that this is the previous result with $m = r = n$.

The following is a useful fact for rewriting product of binomial coefficients.

$$\boxed{\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.}$$

Algebraic proof: Plugging in the definitions for the binomial coefficients we have

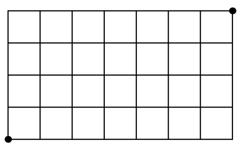
$$\begin{aligned}
\binom{n}{m}\binom{n-m}{k-m} &= \frac{n!}{m!(n-m)!}\,\frac{(n-m)!}{(k-m)!(n-k)!} \\
&= \frac{n!}{(n-k)!m!(k-m)!} = \frac{n!}{k!(n-k)!}\,\frac{k!}{m!(k-m)!} \\
&= \binom{n}{k}\binom{k}{m}.
\end{aligned}$$ □

Combinatorial proof: Examining the left hand side we first pick k out of n elements and then we pick m out of k elements. We can interpret this as the number of sets A and B so that $A \subseteq B \subseteq X$ with $|A| = m$, $|B| = k$ and $|X| = n$. That is, given an n element set X we first pick B as a k element subset of X and then pick A as an m element subset of B.

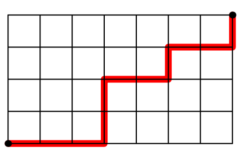
We could also count this in a different way. Namely, we first pick the set A (which can be done in $\binom{n}{m}$ ways and then pick B so that $A \subseteq B \subseteq X$. To do the second step we need to "fill out" the set B by adding an additional $k - m$ elements to the k already chosen for A, since there are $n-m$ elements not in A to choose from we can do this in $\binom{n-m}{k-m}$ ways. □

We have already seen how to use committees to give identities about binomial coefficients. Another useful object is studying walks on a square lattice.

Example: On the square lattice shown below how many different walks are there from $(0,0)$ to $(7,4)$ which consists of steps to the right by one unit or up one unit?

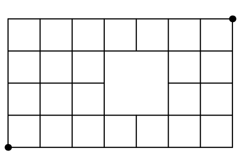


Solution: Looking at the walks we see that we will need to take 7 steps to the right and 4 steps up, for a total of 11 steps. Moreover, every walk from $(0,0)$ to $(7,4)$ can be encoded as a series of right steps (R) and up steps (U). For instance the path $RRRUURRURRU$ corresponds to the following path.



So the number of walks is the same as the number of ways to arrange seven Rs and four Us which is $\binom{11}{4} = \binom{11}{7} = 330$ (i.e., choose when we take an up step or choose when we take a right step).

Example: On the square lattice shown below how many different walks are there from $(0,0)$ to $(7,4)$ which consists of steps to the right by one unit or up one unit?



Solution: This problem is nearly the exact same as the problem before except now we have to forbid some walks. In particular we have to throw out all walks that pass through the point $(4,2)$. So let us count how many walks there are that pass through $(4,2)$. Such a walk can be broken into two parts, namely a walk from $(0,0)$ to $(4,2)$ (of which there are $\binom{6}{2}$ such walks) and a walk from $(4,2)$ to $(7,4)$ (of which there are $\binom{5}{2}$ such walks). Since we can combine these two halves of the walk arbitrarily then by the Rule of Multiplication the number of walks that pass through $(4,2)$ is $\binom{6}{2}\binom{5}{2} = 150$. Therefore the number of walks not passing through $\binom{4}{2}$ is $330 - 150 = 180$.

Lecture 6 – April 10

As another application of the binomial theorem we have the following.

For $n \geq 1$

$$\sum_{k=0}^{n} (-1)^n \binom{n}{k} = 0.$$

In other words starting with the second row and going down if we sum along the rows alternating sign as we go the result is 0. Given the symmetry $\binom{n}{k} = \binom{n}{n-k}$ it trivially holds when k is odd, it is not trivial to show that it holds when k is even.

Algebraic proof: In the binomial theorem set $x = -1$ and $y = 1$ to get

$$0 = (-1+1)^n = \sum_{k=0}^{n} (-1)^k 1^{n-k} \binom{n}{k} = \sum_{k=0}^{n} 1^k \binom{n}{k}. \qquad \square$$

Combinatorial proof: First note that this is equivalent to showing the following:

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

The left hand side counts the number of subsets of $\{1, 2, \ldots, n\}$ with an *even* number of elements while the right hand side counts the number of subsets with an *odd* number of elements. To show that we have an equal number of these two types we will use an *involution argument.*

An involution on a set X is a mapping $\phi : X \to X$ so that $\phi(\phi(x)) = x$. Given an involution ϕ the elements then naturally split into fixed points (elements with $\phi(x) = x$) or pairs (two elements $x \neq y$ where $\phi(x) = y$ and $\phi(y) = x$).

In our case our involution will act on $2^{[n]}$ (the set of all subsets of $\{1, 2, \ldots, n\}$) and is defined for a subset A as follows:

$$\phi(A) = \begin{cases} A \setminus \{1\} & \text{if 1 is in } A, \\ A \cup \{1\} & \text{if 1 is not in } A. \end{cases}$$

In other words we take a set and if 1 is in it we remove it and if 1 is not in it we add it. We now make some observations. First, for any subset A we have $\phi(\phi(A)) = A$ so that it is an involution. Further, there are no fixed points since 1 must either be in or not in a set. So we can now break the collection of subsets into pairs $\{A, \phi(A)\}$. Finally, by the involution A and $\phi(A)$ will differ by exactly one element so one of them is even and one of them is odd. So we now have a way to pair every subset with an even number of elements with a subset with an odd number of elements. So the number of such subsets must be equal. $\square$

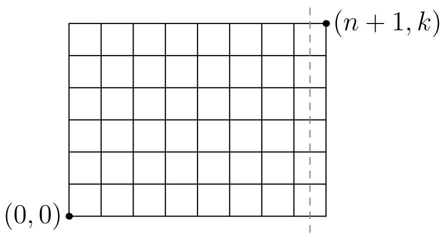
We now give two identities which are useful for simplifying sums of binomial coefficients.

$$\binom{n}{0}+\binom{n+1}{1}+\cdots+\binom{n+k}{k}=\binom{n+k+1}{k},$$

and

$$\binom{k}{k}+\binom{k+1}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1}.$$

To prove the first one we will count the number of walks from $(0,0)$ to $(n+1,k)$ using right steps and up steps in two ways.



We must make $n+1$ steps to the right and k steps up. So the number of ways to walk from one corner to the other is the number of ways to choose when to make the up steps which is

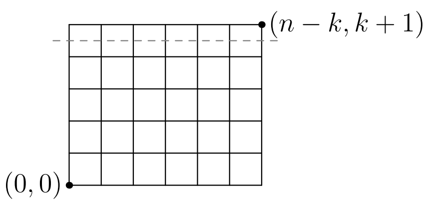
$$\binom{n+k+1}{k}.$$

We can also count the number of walks by grouping them according to which line segment is used in the last step to the right (in the picture above this corresponds to grouping according to the line segment which intersects the dotted line). These line segments go from (n,i) to $(n+1,i)$ where $i=0,\ldots,k$. Once we have crossed the line segment there is only one way to finish the walk to the corner (straight up the side). On the other hand the number of ways to get to (n,i) is $\binom{n+i}{i}$. So by the rule of addition we have that the total number of walks is

$$\binom{n+0}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\cdots+\binom{n+k}{k}.$$

Combining these two different ways to count the paths gives the identity.

A proof of the other result can be done similarly using the following picture (we leave it to the interested reader to fill in the details).



These identities can also be proved by using $\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$. For example, we have the following.

$$\begin{aligned}\binom{n+1}{k+1} &= \binom{n}{k}+\binom{n}{k+1}\\ &= \binom{n}{k}+\binom{n-1}{k}+\binom{n-1}{k+1}\\ &= \cdots\\ &= \binom{n}{k}+\binom{n-1}{k}+\cdots+\binom{k+1}{k}+\binom{k+1}{k+1}\\ &= \binom{n}{k}+\binom{n-1}{k}+\cdots+\binom{k+1}{k}+\binom{k}{k}.\end{aligned}$$

When I was taught these identities they were called the "hockey stick" identities. This name comes from the pattern that they form in Pascal's triangle. For instance we have that $\binom{6}{3}$ (shown below in blue) is the $\binom{2}{2}+\cdots+\binom{5}{2}$ (shown below in red).

$$\begin{array}{c}\binom{0}{0}\\ \binom{1}{0}\ \binom{1}{1}\\ \binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}\\ \binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}\\ \binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}\\ \binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}\\ \binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6}\end{array}$$

The other identity corresponds to looking at the mirror image of Pascal's triangle.

We now give an application of the hockey stick identity.

$$\sum_{i=0}^{n} i^2=\frac{n(n+1)(2n+1)}{6}$$

In the first lecture we saw how to add up the sum of i by counting dots in two different ways. Here we want to sum up i^2, the problem is that we don't have an easy method to do that. We *do* have an easy method to add up the binomial coefficients, so if we can rewrite i^2 in terms of binomial coefficients then we can easily answer this question. So consider the following.

$$i^2=2\frac{i(i-1)}{2}+i=2\binom{i}{2}+\binom{i}{1}.$$

(The ability to rewrite i^2 as a combination of binomial coefficients can also be used for any other polynomial expression of i. The trick is to find the coefficients involved. For the case of polynomials of the form i^ℓ we have that the coefficient of $\binom{i}{k}$ is $k!\left\{{\ell \atop k}\right\}$ where $\left\{{\ell \atop k}\right\}$

are the Stirling numbers of the second kind, which we will discuss in a later lecture.)

Now using this new way to write i^2 we have

$$\begin{aligned}\sum_{i=0}^{n} i^2 &= \sum_{i=0}^{n}\left(2\binom{i}{2}+\binom{i}{1}\right)\\ &= 2\sum_{i=0}^{n}\binom{i}{2}+\sum_{i=0}^{n}\binom{i}{1}\\ &= 2\binom{n+1}{3}+\binom{n+1}{2}\\ &= 2\frac{(n+1)n(n-1)}{6}+\frac{(n+1)n}{2}\\ &= \frac{(n+1)n}{6}\big(2(n-1)+3\big)\\ &= \frac{(n+1)n(2n+1)}{6}.\end{aligned}$$

The only difficult step now is going from the second to the third line, which is done using the hockey stick identity.

There is a more general form of the binomial theorem known as the multinomial theorem ("multinomial" referring to many variables as compared to "binomial" which refers to two). It states

$$\begin{aligned}&(x_1+x_2+\cdots+x_k)^n\\ &= \sum_{\substack{n_1+n_2+\cdots+n_k=n\\ n_1\geq 0, n_2\geq 0,\ldots,n_k\geq 0}} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k},\end{aligned}$$

where

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}$$

is the multinomial coefficient we have seen in previous lectures.

Returning to the binomial theorem, we have that

$$(1+x)^n = \binom{n}{0}+\binom{n}{1}x+\cdots+\binom{n}{n}x^n.$$

In particular for n fixed all of the binomial coefficients $\binom{n}{k}$ are "stored" as coefficients inside of the function $f(x)=(1+x)^n$. In general given any sequence of numbers $a_0, a_1, a_2, \ldots$ (possibly infinitely many) we can store these as coefficients of a function which we call the *generating function* as follows

$$g(x) = a_0 + a_1x + a_2x^2 + \cdots .$$

(There are different ways that we can store the sequence as coefficients leading to different classes of generating functions. This method is what is known as an ordinary generating function, in a later lecture we will see what are known as exponential generating functions.)

Generally speaking we will treat the function as a "formal" series, i.e., as an algebraic rather than analytic structure. This makes it convenient to work with when we do not need to worry about convergence. Nevertheless we will make extensive use of analytic techniques, when we do that is when we will start to worry about convergence.

Let us now consider the problem of finding the generating function for the sequence $a_n = \binom{n}{k}$ where k is fixed. In other words we want to find a function so that its series expansion (note that there are infinitely many a_n in this case) is

$$g(y) = \sum_{n}\binom{n}{k}y^n.$$

We can do this by actually investigating a much more general function. Namely we will use a generating function with two variables x and y and is defined as follows:

$$h(x,y) = \sum_{k,n}\binom{n}{k}x^ky^n.$$

Using the binomial theorem we have

$$\begin{aligned}h(x,y) &= \sum_{k,n}\binom{n}{k}x^ky^n\\ &= \sum_{n}y^n\left(\sum_{k}\binom{n}{k}x^k\right)\\ &= \sum_{n}y^n(1+x)^n \;=\; \sum_{n}\big(y(1+x)\big)^n\\ &= \frac{1}{1-y(1+x)} \;=\; \frac{1}{1-y-xy}.\end{aligned}$$

(Note that this function stores the *entire* Pascal's triangle inside of it.)

In the above derivation we used the following important identity,

$$1+z+z^2+\cdots = \sum_{k=0}^{\infty}z^k = \frac{1}{1-z}.$$

Analytically this is true for $|z|<1$, formally this is always true.

Now to find our desired $g(y)$ we are looking for the coefficient of x^k. So we now rewrite $h(x,y)$ as a power series in x and get

$$\begin{aligned}h(x,y) &= \frac{1}{1-y-xy} \;=\; \frac{1}{1-y}\frac{1}{1-\frac{y}{1-y}x}\\ &= \frac{1}{1-y}\sum_{k=0}^{\infty}\left(\frac{y}{1-y}x\right)^k\\ &= \sum_{k=0}^{\infty}\left(\frac{y^k}{(1-y)^{k+1}}\right)x^k.\end{aligned}$$

Finally, reading off the coefficient for x^k in $h(x, y)$ gives us our desired function, namely

$$g(y) = \sum_n \binom{n}{k} y^n = \frac{y^k}{(1-y)^{k+1}}.$$

Lecture 7 – April 13

Today we look at one motivation for studying generating functions, namely a connection between polynomials and distribution problems. Let us start with a simple example.

Example: Give an interpretation for the coefficient of $x^1 1$ for the polynomial

$$g(x) = (1 + x^2 + x^3 + x^5)^3.$$

Solution: First note that we can write $g(x)$ as

$$(1 + x^2 + x^3 + x^5)(1 + x^2 + x^3 + x^5)(1 + x^2 + x^3 + x^5)$$

and that a typical term which we will get when we expand has the form $x^{n_1}x^{n_2}x^{n_3}$ where the term x^{n_1} comes from picking a term from the first polynomial (so $n_1 \in \{0, 2, 3, 5\}$), the term x^{n_2} comes from picking a term from the second polynomial (so $n_2 \in \{0, 2, 3, 5\}$), and the term x^{n_3} comes from picking a term from the third polynomial (so $n_3 \in \{0, 2, 3, 5\}$). Since we are interested in the coefficient of x^{11} then we need $n_1 + n_2 + n_3 = 11$.

So this can be rephrased as a balls and bins problem where we are trying to distribute 11 balls among three bins where in each bin we can have either 0, 2, 3 or 5 balls.

Example: Give an interpretation for the coefficient of x^r for the function

$$g(x) = (1+x+x^2)(1+x+x^2+\cdots)(x+x^3+x^5+\cdots).$$

Solution: By the same reasoning as we have before a typical term in the expansion looks like $x^{n_1+n_2+n_3}$ with n_1 being 0, 1 or 2; $n_2 \geq 0$; and $n_3 \geq 0$ and even.

So this can be rephrased as a balls and bins problem where we are trying to distribute r balls among three bins where in the first bin we put 0, 1 or 2 balls; in the second bin we put any number of balls we want; in the third bin we put in an odd number of balls.

In general given a polynomial

$$g(x) = g_1(x)g_2(x)\cdots g_k(x)$$

with each g_i a sum of some power of xs, i.e.,

$$g_i(x) = \sum_\ell x^{q_\ell^i}.$$

Then the coefficient of x^r in $g(x)$ is the number of ways to place r balls into k bins so that in the ith bin the number of balls is q_ℓ^i for some ℓ. (The idea is that the term $g_i(x)$ tells you the restrictions about the kind of balls that can be placed into that bin. It is important that the coefficients of the $g_i(x)$ are all 1's.)

We can also start with a balls and bins problem and translate it into finding the coefficient of some appropriate polynomial.

Example: Translate the following problem into finding a coefficient of some function: "How many ways are there to put 12 balls into four bins so that the first two bins have between 2 and 5 balls, the third bin has at least 3 balls and the last bin has an odd number of balls?"

Solution: Since there will be 12 balls we will be looking for the coefficient of x^{12}. Now let us translate the condition on each bin. For the first two bins there are between 2 and 5 balls so

$$g_1(x) = g_2(x) = x^2 + x^3 + x^4 + x^5$$

(remember, the powers list the number of balls that can be put in the bin). For the third bin we have to have at least 3 balls so

$$g_3(x) = x^3 + x^4 + \cdots.$$

We could also use the function

$$g_3'(x) = x^3 + x^4 + \cdots + x^{12},$$

the difference being that we stop and have a polynomial instead of an infinite series. The reason we can do this is that we are only interested in the coefficient of x^{12}, anything which will not contribute to that coefficient can be dropped without changing the answer. However by keeping the series we can use the same function to answer the question for any arbitrary number of balls and so the first option is more general. Finally for the last bin we have

$$g_4(x) = x + x^3 + x^5 + \cdots.$$

Putting it altogether we are trying to find the coefficient of x^{12} for the function

$$(x^2+x^3+x^4+x^5)^2(x^3+x^4+\cdots)(x+x^3+x^5+\cdots).$$

Of course translating one problem to another is only good if we have a technique to solve the second problem. So our next natural step is to find ways to determine the coefficient of functions of this type. To help us do this we will make use of the following identities.

$$
\begin{aligned}
(1+x)^n &= \sum_{k=0}^{n}\binom{n}{k}x^k \\
(1-x^m)^n &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}x^{mk} \\
\frac{1-x^{m+1}}{1-x} &= 1+x+x^2+\cdots+x^m \\
\frac{1}{1-x} &= 1+x+x^2+\cdots \qquad \text{(for } |x|<1\text{)} \\
\frac{1}{(1-x)^n} &= \sum_{k\geq 0}\binom{n-1+k}{k}x^k
\end{aligned}
$$

The first two are simple applications of the binomial theorem. The third is easily verifiable by multiplying both sides by $(1-x)$ and simplifying. The fourth one is a well known sum and can be considered the limiting case of the third one. (Note that as a function of x, i.e., analytically, this makes sense only when $|x|<1$. Formally there is no restriction on x, this is because formally x^k is acting as a placeholder.)

To see why the last one holds we note that this is

$$
\begin{aligned}
\frac{1}{(1-x)^n} &= \underbrace{\frac{1}{1-x}\frac{1}{1-x}\cdots\frac{1}{1-x}}_{n\text{ times}} \\
&= \underbrace{(1+x+x^2+\cdots)\cdots(1+x+x^2+\cdots)}_{n\text{ times}}
\end{aligned}
$$

Translating this into a balls and bins problem, the coefficient of x^k would correspond to the number of solutions of

$$e_1+e_2+\cdots+e_n=k$$

where each $e_i\geq 0$. We have already solved this problem using bars and stars, namely this can be done in $\binom{n-1+k}{k}$ ways, giving us the desired result.

We will need one more tool, and that is a way to multiply functions. We have the following which is a simple exercise in expanding and grouping coefficients.

> Given
>
> $$
> \begin{aligned}
> f(x) &= a_0+a_1x+a_2x^2+\cdots, \\
> g(x) &= b_0+b_1x+b_2x^2+\cdots.
> \end{aligned}
> $$
>
> Then
>
> $$
> \begin{aligned}
> f(x)g(x) = a_0b_0 &+ (a_0b_1+a_1b_0)x+\cdots \\
> &+ (a_0b_k+a_1b_{k-1}+\cdots+a_kb_0)x^k+\cdots.
> \end{aligned}
> $$

The key is that the coefficient of x^k is found by combining elements $(a_ix^i)(b_{k-i}x^{k-1})$ for $i=0,\ldots,k$.

We now show how to use these various rules to solve a combinatorial problem.

Example: How many ways are there to put 25 balls into 7 boxes so that each box has between 2 and 6 balls?

Solution: First we translate this into a polynomial problem. We are looking for 25 balls so we will be looking for a coefficient of x^{25}. The constraint on each box is the same, namely the between 2 and 6 balls so that the function we will be considering is

$$
\begin{aligned}
g(x) &= (x^2+x^3+x^4+x^5+x^6)^7 \\
&= x^{14}(1+x+x^2+x^3+x^4)^7.
\end{aligned}
$$

Here we pulled out an x^2 out of the inside which then becomes x^{14} in front. Now since we are looking for the coefficient of x^{25} and we have a factor of x^{14} in front, our problem is equivalent to finding the coefficient of x^{11} in

$$g'(x)=(1+x+x^2+x^3+x^4)^7.$$

(Combinatorially, this is also what we would do. Namely distribute two balls into each bin, fourteen balls in total, and then decide how to deal with the remaining 11.)

Using our identities we have

$$
\begin{aligned}
&g'(x)=(1+x+x^2+x^3+x^4)^7 \\
&=\left(\frac{1-x^5}{1-x}\right)^7=(1-x^5)^7\cdot\frac{1}{(1-x)^7} \\
&=\left(\binom{7}{0}-\binom{7}{1}x^5+\binom{7}{2}x^{10}+\cdots\right)\left(\sum_{k\geq 0}\binom{6+k}{k}x^k\right)
\end{aligned}
$$

Finally, we use the rule for multiplying functions together to find the coefficient of x^{11}. In particular note that in the first part of the product only three terms can be used, as all the rest will not contribute to the coefficient of x^{11}. So multiplying we have that the coefficient to x^{11} is

$$\binom{7}{0}\binom{6+11}{11}-\binom{7}{1}\binom{6+6}{6}+\binom{7}{2}\binom{6+1}{1}=6,055.$$

This can also be done combinatorially. The first term counts the numbers without restriction. The second term then takes off the exceptional cases, but it takes off too much and so the third term is there to add it back in.

Lecture 8 – April 15

We can use the rule for multiplying functions together to prove various results. For instance, let $f(x) = (1+x)^m$ and $g(x) = (1+x)^n$, then $f(x)g(x) = (1+x)^{m+n}$. We now compute the coefficient of x^r of $f(x)g(x)$ in two different ways to get

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}.$$

The left hand side follows by using the rule of multiplying $f(x)$ and $g(x)$ while the right hand side is the binomial theorem for $f(x)g(x)$.

We turn to a problem of finding a generating function.

Example: Show that the generating function for the number of integer solutions to

$$e_1 + e_2 + e_3 + e_4 = n$$

with $0 \le e_1 \le e_2 \le e_3 \le e_4$ is

$$f(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}.$$

Solution: First note that we can rewrite the function $f(x)$ as

$$f(x) = (1+x+x^2+x^3+\cdots)(1+x^2+x^4+x^6+\cdots) \\ \times (1+x^3+x^6+x^9+\cdots)(1+x^4+x^8+x^{12}+\cdots).$$

Note that translating this back into a balls and bins problem this says that we have four bins. In the first bin we have any number of balls, in the second bin we have a multiple of two number of balls, in the third bin we have a multiple of three number of balls and in the fourth bin we have a multiple of four number of balls. In other words the generating function $f(x)$ counts the number of solutions to

$$f_1 + 2f_2 + 3f_3 + 4f_4 = n$$

with $f_1, f_2, f_3, f_4 \ge 0$. We now need to show that these two problems have the same number of solutions. To do this, let us start with a solution of $e_1+e_2+e_3+e_4 = n$ and pictorially represent our solution by putting four row of $*$s with e_4 stars in the first row, e_3 stars in the second row, e_2 stars in the third row and e_1 stars in the fourth row. Finally, let f_4 be the number of columns with four $*$s, f_3 the number of columns with three $*$s, f_2 the number of columns with two $*$s and f_1 the number of columns with one $*$. This gives a solution to $f_1 + 2f_2 + 3f_3 + 4f_4 = n$. This gives a one-to-one correspondence between solutions to the two different problems, so they have an equal number of solutions giving the result.

As an example of the last step, suppose we start with

$$3 + 5 + 9 + 11 = 28.$$

Then pictorially this corresponds to the following picture (we have marked the number of $*$s in each row/column).

11	*	*	*	*	*	*	*	*	*	*	*
9	*	*	*	*	*	*	*	*	*		
5	*	*	*	*	*						
3	*	*	*								
	4	4	4	3	3	2	2	2	2	1	1

So translating this gives us the solution

$$(2) + 2(4) + 3(2) + 4(3) = 28$$

to the second problem.

This problem says that the number of ways to break n into a sum of at most four parts is the same as the number of ways to break n into a sum of parts where each part has size at most four. We will generalize this idea in the next lecture. For now we start by looking at partitions. A partition of a number n is a way to break n up as a sum of smaller pieces. For instance

$$1+1+1+1 = 1+1+2 = 1+3 = 2+2 = 4$$

are the ways to break four into pieces. We do not care about the order of the pieces so that $1+1+2$ and $1+2+1$ and $2+1+1$ are considered the same partition. This corresponds to the number of ways to distribute identical balls among identical bins (for now we will suppose that we have an unlimited number of bins so that we can have any number of parts in the partition).

Let $p(n)$ denote the number of partitions of n.

n	partitions of n	$p(n)$
0	0	1
1	1	1
2	$1+1,\ \ 2$	2
3	$1+1+1,\ \ 1+2,\ \ 3$	3
4	$1+1+1+1,\ \ 1+1+2,\ \ 1+3$ $2+2,\ \ 4$	5
5	$1+1+1+1+1,\ \ 1+1+1+2$ $1+1+3,\ \ 1+2+2,\ \ 1+4$ $2+3,\ \ 5$	7
6	$1+1+1+1+1+1$ $1+1+1+1+2,\ \ 1+1+1+3$ $1+1+2+2,\ \ 1+1+4,\ \ 1+2+3$ $1+5,\ \ 2+2+2,\ \ 2+4,\ \ 3+3,\ \ 6$	11

The function $p(n)$ has been well studied, but is highly non-trivial. One of the greatest mathematical geniuses of the twentieth century was the Indian mathematician Srinivasa Ramanujan. Originally a clerk in India he was able through personal study discover dozens of previously unknown relationships about partitions. He sent these (along with other discoveries) to mathematicians in England and one of them, G. H. Hardy, was able to recognize the writings as something new and exciting which caused him to bring Ramanujan to England which resulted in one of the great mathematical collaborations of the twentieth century. Examples of facts that Ramanujan discovered include that $p(5n+4)$ is divisible by 5; $p(7n+5)$ is divisible by 7; and $p(11n+6)$ is divisible by 11.

One natural question is to whether the exact value of $p(n)$ is known. There is a "nice" formula for $p(n)$, namely

$$p(n)=\frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty}A_k(n)\sqrt{k}\left[\frac{d}{dx}\frac{\sinh\left(\frac{\pi}{k}\sqrt{x-\frac{1}{24}}\right)}{\sqrt{x-\frac{1}{24}}}\right]_{x=n}$$

where $A_k(n)$ is a specific sum of $24k$th roots of unity. Proving this is beyond the scope of our class!

We now turn to the (much) easier problem of finding the generating function for $p(n)$, that is we want to find

$$\sum_{n\geq 0}p(n)x^n.$$

The first thing is to observe that a partition of size n corresponds to a solution of

$$e_1+2e_2+3e_3+\cdots+ke_k+\cdots=n,$$

where each e_i represents the number of parts of size i. Notice in particular the the contribution of e_k will be a multiple of k. So turning this back into balls and bins we have n balls and infinitely many bins. So the number of solutions (using what we did last time) is

$$\sum_{n\geq 0}p(n)x^n=\underbrace{(1+x+x^2+x^3+\cdots)}_{e_1\text{ term}}\times\underbrace{(1+x^2+x^4+x^6+\cdots)}_{e_2\text{ term}}\times\underbrace{(1+x^3+x^6+x^9+\cdots)}_{e_3\text{ term}}\times\cdots\times\underbrace{(1+x^k+x^{2k}+x^{3k}+\cdots)}_{e_k\text{ term}}\times\cdots.$$

Since

$$1+z^k+z^{2k}+z^{3k}+\cdots=\frac{1}{1-z^k},$$

we can rewrite the generating function as

$$\begin{aligned}\sum_{n\geq 0}p(n)x^n &= \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}\cdots\frac{1}{1-x^k}\cdots\\ &= \prod_{k\geq 1}\frac{1}{1-x^k}.\end{aligned}$$

We can do variations on this. For instance we can look at partitions of n where no part is repeated. In other words we want to restrict our partitions so that $e_k=0$ or 1 for each k. If we denote these partitions by $p_d(n)$ then

$$\begin{aligned}\sum_{n\geq 0}p_d(n)x^n &= (1+x)(1+x^2)\cdots(1+x^k)\cdots\\ &= \prod_{k\geq 1}(1+x^k)\end{aligned}$$

Similarly we can count partitions with each part odd. In other words we want to restrict out partitions so that $e_{2k}=0$ for each k. If we denote these partitions by $p_o(n)$ then it is easy to modify our construction for $p(n)$ to get

$$\sum_{n\geq 0}p_o(n)=\prod_{k\geq 1}\frac{1}{1-x^{2k-1}}.$$

We can combine these two generating functions to derive a remarkable result.

For $n\geq 1$ we have $p_d(n)=p_o(n)$.

Remarkably, we do not know in general what $p_d(n)$ or $p_o(n)$ is, nevertheless we do know that they are equal! As an example of this we have that $p_o(6)=4$ because of the partitions

$$1+1+1+1+1+1,\quad 1+1+1+3,\quad 1+5,\quad 3+3,$$

while $p_d(6)=4$ because of the partitions

$$1+2+3,\quad 1+5,\quad 2+4,\quad 6$$

It suffices for us to show that the two generating functions are identical. If the functions are the same then the coefficients must be the same giving us the desired result. So we have that

$$\begin{aligned}\sum_{n\geq 0}p_d(n)x^n &= \prod_{k\geq 1}(1+x^k)\\ &= \prod_{k\geq 1}\frac{1-x^{2k}}{1-x^k}\\ &= \frac{1-x^2}{1-x}\cdot\frac{1-x^4}{1-x^2}\cdot\frac{1-x^6}{1-x^3}\cdot\frac{1-x^8}{1-x^3}\cdots\\ &= \frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdots\\ &= \prod_{k\geq 1}\frac{1}{1-x^{2k-1}} = \sum_{n\geq 0}p_o(n)x^n.\end{aligned}$$

We can also give a bijective proof. To do this we need to give a way to take a partition with odd parts and produce a (unique) partition with distinct parts, and vice versa. This can be done by repeatedly applying the following rule until it cannot be done anymore:

- With the current partition, if any two parts are equal combine them into a single part.

For example for the partition

$$\begin{aligned} 1+1+1+1+1+1 &\rightarrow 2+1+1+1+1 \\ &\rightarrow 2+2+1+1 \\ &\rightarrow 4+1+1 \\ &\rightarrow 4+2 \end{aligned}$$

This rule takes a partition with odd parts and produces a partition with distinct parts. To go in the opposite direction this cane be done by applying the following rule until it cannot be done anymore:

- With the current partition, if any part is even then split it into two equal parts.

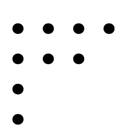
For example for the partition

$$\begin{aligned} 3+4+6 &\rightarrow 2+2+3+6 \\ &\rightarrow 1+1+2+3+6 \\ &\rightarrow 1+1+1+1+3+6 \\ &\rightarrow 1+1+1+1+3+3+3 \end{aligned}$$

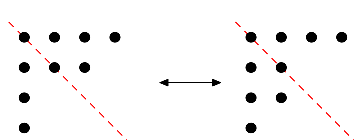
These two rules give a bijective relationship between partitions with an odd number of parts and partitions with distinct parts.

Lecture 9 – April 17

We can visually represent a partition as a series of rows of dots (much like we did in the example in the previous lecture). This representation is known as a *Ferrer's diagram*. The number of dots in each row is the size of the part and we arrange the rows in weakly decreasing order. The partition $4+3+1+1$ would be represented by the following diagram.

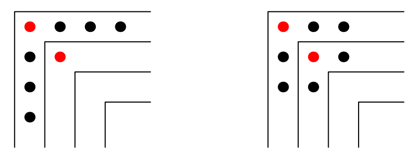


We can use Ferrer's diagrams to gain insight into properties of partitions. For instance one simple operation that we can do is to take the transpose (or the conjugate) of the partition. This is done by "flipping" across the line in the picture below, so that the columns become rows and rows become columns.



Since we do not change the number of dots by taking the transpose this takes a partition and makes a new partition. So in this case we get the new partition $4+2+2+1$.

Note that if we take the transpose of the transpose that we get back the original diagram. There are some partitions such that the transpose of the partition gives back the partition, such partitions are called self conjugate. For instance there are two self conjugate partitions for $n=8$, namely $4+2+1+1$ and $3+3+2$. A famous result says that the number of self conjugate partitions is equal to the number of partitions with distinct odd parts. So for example for $n=8$ there are two partitions with distinct odd parts, namely $7+1$ and $5+3$. One proof of this relationship is based on using Ferrer's diagrams. We will not fill in all of the details here but give the following "hint" for $n=8$.



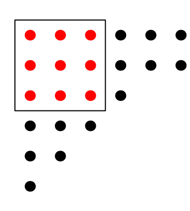
Also note that when we take the transposition that the number of rows becomes the size of the largest part in the transposition while the size of the largest part becomes the number of rows. So mapping a partition to its transpose establishes the following fact.

> There is a one-to-one correspondence between the number of partitions of n into exactly m parts and the number of partitions of n into parts of size at most m with at least one part of size m.

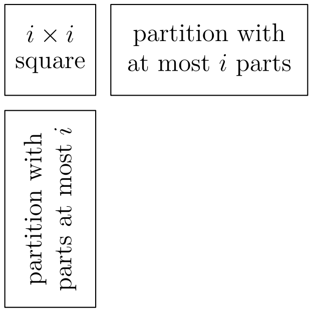
It is easy to count partitions with parts of size at most m with at least one part of size m. In particular if we let $\left|{n \atop m}\right|$ denote the number of partitions of n into exactly m parts then we have (using the techniques from the last lecture)

$$\sum_{n\geq 0} \left|{n \atop m}\right| x^n = \frac{x^m}{(1-x)(1-x^2)\cdots(1-x^m)}.$$

Another object associated with a Ferrer's diagram is the *Durfee square* which is the largest square of dots that is in the upper left corner of the Ferrer's diagram. For example for the partition $6+6+4+3+2+1+1$ we have a 3×3 Durfee square as shown in the following picture.



It should be noted that *every* partition can be decomposed into three parts. Namely a Durfee square and then two partitions, one on the right of the Durfee square and one below the Durfee square as illustrated below.



If we group partitions according to the size of the largest Durfee square and then use generating functions for partitions with at most i parts and partitions with parts at most i (which by transposition are the same generating function) we get the following identity:

$$\sum_{n\geq 0} p(n)x^n = \sum_{n\geq 0}\Bigg(\underbrace{x^{n^2}}_{\text{square}}\underbrace{\prod_{k=1}^{n}\frac{1}{1-x^i}}_{\substack{\text{left}\\\text{partition}}}\underbrace{\prod_{k=1}^{n}\frac{1}{1-x^i}}_{\substack{\text{bottom}\\\text{partition}}}\Bigg)$$

$$= \sum_{n\geq 0}\frac{x^{n^2}}{(1-x)^2(1-x^2)^2\cdots(1-x^n)^2}.$$

There are a lot more fascinating and interesting things that we can talk about in regards to partitions. We finish our discussion with the following.

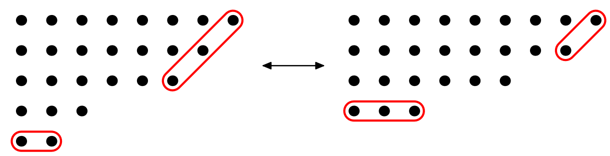
$$\prod_{n\geq 1}(1-x^i) = 1-z-z^2+z^5+z^7-z^{12}-z^{15}+\cdots$$
$$= \sum_{-\infty<j<\infty}(-1)^i z^{(3j^2+j)/2}$$

This is very similar to what we encountered in the previous lecture, namely $\prod(1+x^i)$. We saw that this last expression was used to count the number of partitions with distinct parts. In the current expression something similar is going on, the difference is that previously every partition of n with distinct parts contributed 1 to the coefficient of x^n; now the contribution of a partition of n with distinct parts depends on whether the number of parts is even or odd. Namely, if the number of parts is even then the contribution is 1 and if the number of parts is odd the contribution is -1. In particular, it is not hard to see that

$$\prod_{n\geq 1}(1-x^i) = \sum_{n\geq 0}\big(E(n)-O(n)\big)x^n,$$

where $E(n)$ is the number of partitions of n into an even number of distinct parts and $O(n)$ is the number of partitions of n into an odd number of distinct parts. So to determine the product we need to determine $E(n)-O(n)$, we will see that this value is -1, 0 or 1. To do this we give a bijection between the number of partitions of n into an even number of distinct parts and the number of partitions of n into an odd number of distinct parts. This is done by taking the Ferrer's diagram of a partition into distinct parts and comparing the size of the smallest part (call it q) and the size of the largest 45° run in the upper left corner (call it r). If $q>r$ then take the points from the 45° run and make it into a new part. If $q\geq r$ then take the smallest part and put it in at the end as the 45° run.

An example of what this is doing is shown below. In the partition on the left we take the smallest part and put it at the end of the first few rows while in the partition on the right we take the small 45° run at the end of the first few rows and turn it into the smallest part.



In particular this gives an involution between partitions of n into an odd number of distinct parts and partitions of n into an even number of distinct parts. All that remains is to find the fixed points of the involution, namely those partitions into distinct parts where this operation fails. It is not too hard to see that the only way that this can fail is if we have one of the two following types of partitions.

In the one on the left we have $q=r$ but we cannot move q down to the 45° line because of the overlap. In the one on the right we have $q<r$ but if we take the points on the 45° line it will create a partition which does not have distinct parts. Counting these exceptional cases are partitions of size

$$\frac{3j^2\pm j}{2}.$$

Putting all of this together (along with a little bit of playing with terms) explains the form of the product.

This is known as Euler's pentagonal theorem because the numbers that come up in the powers with nonzero coefficients are pentagonal numbers (which can be formed by stacking a square on top of a triangle).

The pentagonal theorem has a nice application which is useful for quickly generating the number of partitions of n.

We have

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \cdots$$

where the terms on the right hand side are those from the pentagonal theorem.

This follows by noting that

$$\begin{aligned} 1 &= \Big(\prod_{n\geq 0} \frac{1}{1-x^n}\Big)\Big(\prod_{n\geq 1}(1-x^n\Big) \\ &= \Big(\sum_{n\geq 0} p(n)x^n\Big)\Big(\sum_{-\infty<j<\infty}(-1)^j x^{(3j^2+j)/2}\Big) \end{aligned}$$

Now using the rule for multiplying functions together and comparing coefficients on the left and right hand sides we see that for $n \geq 1$ that

$$0 = p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) + \cdots,$$

rearranging now gives the desired result.

We now return to generating functions. The type of generating functions that we have encountered are what are called *ordinary generating functions* which are useful in problems involving combinations. But there is another class of generating functions known as *exponential generating functions* which are useful in problems involving arrangements.

Type of generating function	form	used to count
ordinary	$\displaystyle\sum_{n\geq 0} a_n x^n$	combinations
exponential	$\displaystyle\sum_{n\geq 0} a_n \frac{x^n}{n!}$	arrangements

The term "exponential" comes from the fact that if $a_1 = a_2 = \cdots = 1$ then the resulting function is

$$\sum_{n\geq 1} \frac{x^n}{n!} = e^x.$$

A typical problem involving combinations looks at counting the number of nonnegative solutions to an equation of the form

$$n_1 + n_2 + \cdots + n_k = n$$

where we have restrictions on the n_i. In particular for each solution that we find we add 1 to the count for the number of combinations. For arrangements we start with the same basic problem, namely we must first choose what to arrange so we have to have a nonnegative solution to an equation similar to the form

$$n_1 + n_2 + \cdots + n_k = n$$

where again we have restriction on the n_i. But now after we have chosen our terms we still need to arrange (or order) them. From a previous lecture if we have n_1 objects of type 1, n_2 objects of type 2, and so on then the number of ways to arrange them is

$$\frac{n!}{n_1! n_2! \cdots n_k!},$$

so now for each solution to $n_1 + n_2 + \cdots + n_k = n$ this is what will be contributed to the count for the number of arrangements. The exponential function is perfectly set up to count the number of exponential functions.

To see this consider,

$$\Big(\sum_{\ell\in S_1} \frac{x^\ell}{\ell!}\Big)\Big(\sum_{\ell\in S_2} \frac{x^\ell}{\ell!}\Big)\cdots\Big(\sum_{\ell\in S_k} \frac{x^\ell}{\ell!}\Big)$$

where S_i is the possible values for n_i. Then a typical product will be of the form

$$\frac{x^{n_1}}{n_1!} \cdot \frac{x^{n_2}}{n_2!} \cdots \frac{x^{n_k}}{n_k!}$$

if $n_1 + n_2 + \cdots + n_k = n$ then the contribution to $x^n/n!$ is

$$\frac{n!}{n_1! n_2! \cdots n_k!} \frac{x^n}{n!},$$

the number of arrangements, just like we wanted!

Let us look at two related problems to compare the differences of the two types of generating functions.

Example: Find a generating function which counts the number of ways to pick n digits from 0s, 1s and 2s so that there is at least one 0 chosen and an odd number of 1s.

Solution: Using techniques from previous lectures we have that the (ordinary) generating function is

$$\begin{aligned} &(x+x^2+x^3+\cdots)(x+x^3+x^5+\cdots)(1+x+x^2+\cdots) \\ &\qquad = \frac{x}{1-x}\frac{x}{1-x^2}\frac{1}{1-x} = \frac{x^2}{(1-x)^3(1+x)}. \end{aligned}$$

Example: Find a generating function which counts the number of ternary sequences (sequences formed using the digits 0, 1 and 2) of length n with at least one 0 and an odd number of 1s.

Solution: This problem is related to the previous, except that after we pick out which digits we will be using we still need to arrange the digits. So we will use an exponential generating function to count these. In our case we have

$$\begin{aligned} &\big(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\big)\big(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\big)\big(1+x+\frac{x^2}{2!}+\cdots\big) \\ &= (e^x-1)\big(\frac{e^x-e^{-x}}{2}\big)e^x = \frac{1}{2}(e^{3x}-e^{2x}-e^x-1). \end{aligned}$$

Note that the approach is nearly identical for both problems, the only difference in the (initial) setup for the generating functions is the introduction of the factorial terms. So the same techniques and skills that we learned in setting up problems before still holds.

The only real trick is to decide when to use an exponential generating function and when not to use it. In the end it boils down (for now) to whether or not we need to account for ordering in what we choose. If we don't then go with an ordinary generating function, if we do go with an exponential generating function.

Lecture 10 – April 20

In the last lecture we saw how to set up an exponential generating function to solve an arrangement problem. As with ordinary generating functions, if we want to use exponential generating functions we need to be able to find coefficients of the functions that we set up. To do this we list some identities.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{\ell x} = \sum_{k=0}^{\infty} \ell^k \frac{x^k}{k!}$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\sum_{k\geq n} \frac{x^k}{k!} = e^x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} - \cdots - \frac{x^{n-1}}{(n-1)!}$$

Example: Find the number of ternary sequences (sequences formed using the digits 0, 1 and 2) of length n with at least one 0 and an odd number of 1s.

Solution: From last time we know that the exponential generating function $g(x)$ which counts the number of ternary sequences is

$$\begin{aligned} g(x) &= \frac{1}{2}(e^{3x} - e^{2x} - e^x - 1) \\ &= \frac{1}{2}\bigg(\sum_{n\geq 0} 3^n \frac{x^n}{n!} - \sum_{n\geq 0} 2^n \frac{x^n}{n!} - \sum_{n\geq 0} \frac{x^n}{n!} - 1\bigg) \\ &= \sum_{n\geq 1} \frac{1}{2}\big(3^n - 2^n - 1\big)\frac{x^n}{n!}. \end{aligned}$$

So reading off the coefficient of $x^n/n!$ we have that the number of such solutions is $(3^n - 2^n - 1)/2$ for $n \geq 1$ and 0 for $n = 0$.

Example: Find the exponential generating function where the nth coefficient counts the number of n letter words which can be formed using letters from the word "BOOKKEEPER".

Solution: We will use an exponential generating function since we are interested in counting the number of words and the arrangement of letters in words gives different words. Since order is important we will go with an exponential generating function. We group by letters. There are 3 Es, 2 Ks and Os and 1 B, P and R. So putting this together the desired exponential generating function is

$$\begin{aligned} f(x) &= \big(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\big)\big(1 + x + \frac{x^2}{2!}\big)^2\big(1 + x\big)^3 \\ &= 1 + 6x + 33\frac{x^2}{2!} + 166\frac{x^3}{3!} + 758\frac{x^4}{4!} + 3100\frac{x^5}{5!} \\ &\qquad + 11130\frac{x^6}{6!} + 34020\frac{x^7}{7!} + 84000\frac{x^8}{8!} \\ &\qquad + 151200\frac{x^9}{9!} + 151200\frac{x^{10}}{10!}. \end{aligned}$$

(The last step is done by multiplying the polynomial out, letting the computer do all of the heavy lifting. So we can read off the coefficients and see, for example, that there are 34020 words of length 7 that can be formed using the letters from BOOKKEEPER.)

Example: How many ways are there to place n distinct people into three different rooms with at least one person in each room?

Solution: This does not look like an arrangement problem, so we might not automatically think of using exponential generating functions. However, we previously saw that the number of ways of arranging n_1 objects of type 1, n_2 objects of type 2, ..., and n_k objects of type k is the same as the number of ways to distribute n distinct objects into k bins so that bin 1 gets n_1 objects, bin 2 gets n_2 objects, ..., and bin k gets n_k objects.

So this problem is perfect for exponential generating functions. In particular, since there are three room we can set this up as $n_1 + n_2 + n_3 = n$ where n_i are the number people in room i. Since each room must have at least one person we have that $n_i \geq 1$. So the exponential generating function is

$$\begin{aligned} h(x) &= \big(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\big)^3 \\ &= \big(e^x - 1\big)^3 \\ &= e^{3x} - 3e^{2x} + 3e^x - 1 \\ &= \sum_{n\geq 0} 3^n \frac{x^n}{n!} - 3\sum_{n\geq 0} 2^n \frac{x^n}{n!} + 3\sum_{n\geq 0} \frac{x^n}{n!} - 1 \\ &= \sum_{n\geq 1} \big(3^n - 3{\cdot}2^n + 3\big)x^n. \end{aligned}$$

Reading off the coefficients we see that the answer to our question with n people is $3^n - 2{\cdot}2^n + 3$. In a later lecture we will see an alternative way to derive this expression using the principle of inclusion-exclusion.

Lecture 11 – April 24

It frequently happens that we want to prove a statement is true for $n = 1, 2, \ldots$. For instance on the first day we saw that

$$1 + 2 + \cdots = \frac{n(n+1)}{2}, \qquad \text{for } n \geq 1.$$

We gave a proof for this using a method of counting a set of dots in two different ways. However we could prove it in another way. First let us prove it for $n = 1$. Since $1 = 1(1+1)/2$, the case $n = 1$ is true. Now let us assume that the statement is true for n (i.e., we can assume it is true $1 + 2 + \cdots + n = n(n+1)2)$ and consider the case $n + 1$. We have

$$\underbrace{1 + 2 + \cdots + n}_{\text{case for } n} + (n+1) = \frac{n(n+1)}{2} + (n+1)$$
$$= (n+1)\left(\frac{n}{2} + 1\right) = \frac{(n+1)(n+2)}{2}.$$

This shows that the case $n + 1$ is also true. Between these two statements this shows it is true for $n = 1, 2, 3, \ldots$. (I.e., we have shown it is true for $n = 1$, then by the second part since it is true for $n = 1$ it is true for $n = 2$, then again by the second part since it is true for $n = 2$ it is true for $n = 3$, ….)

This is an example of *mathematical induction* which is a useful method of proving statements for $n = 1, 2, \ldots$ (though we don't have to start at 1 we can start at any value). In general we need to prove two parts:

1. The statement holds in the first case.
2. Assuming the statement holds for the cases $k \leq n$, show that the statement also hold for $n + 1$.

(This is what is known as *strong induction.* Usually it will suffice for us to only assume the previous case, i.e., assume the case $k = n$ is true and then show $n+1$ also holds (which is known as *weak induction.*)

The idea behind proof by induction is analogous to the idea of climbing a ladder. We first need to get on the ladder (prove the first case), and then we need to be able to move from one rung of the ladder to the next (show that if one case is true then the next case is true. It is important that you remember to prove the first case!

The *most* important thing about an induction proof is to see how to relate the previous case(s) to the current case. Often once that is understood the rest of the proof is very straightforward.

Example: Prove for $n \geq 1$,

$$-1^2 + 2^2 - 3^2 + \cdots + (-1)^n n^2 = (-1)^n \frac{n(n+1)}{2}.$$

Solution: We first prove the case $n = 1$. Since

$$-1^2 = -1 = (-1)^1 \frac{1(1+1)}{2},$$

the first case holds. Now we assume the case corresponding to n is true, and now we want to prove the case corresponding to $n+1$ is true. That is, we assume

$$-1^2 + 2^2 - 3^2 + \cdots + (-1)^n n^2 = (-1)^n \frac{n(n+1)}{2}.$$

and we want to show

$$-1^2 + 2^2 - 3^2 + \cdots + (-1)^n n^2 + (-1)^{n+1}(n+1)^2$$
$$= (-1)^{n+1} \frac{(n+1)(n+2)}{2}.$$

So let us begin, we have

$$\underbrace{-1^2 + 2^2 - 3^2 + \cdots + (-1)^n n^2}_{\text{case corresponding to } n} + (-1)^{n+1}(n+1)^2$$
$$= (-1)^n \frac{n(n+1)}{2} + (-1)^{n+1}(n+1)^2$$
$$= (-1)^{n+1}(n+1)\left(-\frac{n}{2} + (n+1)\right)$$
$$= (-1)^{n+1} \frac{(n+1)(n+2)}{2}. \qquad \square$$

Example: Prove for $n \geq 1$,

$$\frac{1}{1{\cdot}2} + \frac{1}{2{\cdot}3} + \cdots + \frac{1}{n{\cdot}(n+1)} = \frac{n}{n+1}.$$

Solution: We first prove the case $n = 1$. Since

$$\frac{1}{1{\cdot}2} = \frac{1}{2} = \frac{1}{1+1},$$

the first case holds. Now we assume the case corresponding to n is true, and now we want to prove the case corresponding to $n+1$ is true. That is, we assume

$$\frac{1}{1{\cdot}2} + \frac{1}{2{\cdot}3} + \cdots + \frac{1}{n{\cdot}(n+1)} = \frac{n}{n+1}.$$

and we want to show

$$\frac{1}{1{\cdot}2} + \frac{1}{2{\cdot}3} + \cdots + \frac{1}{n{\cdot}(n+1)} + \frac{1}{(n+1){\cdot}(n+2)} = \frac{n+1}{n+2}.$$

So let us begin, we have

$$\underbrace{\frac{1}{1{\cdot}2} + \frac{1}{2{\cdot}3} + \cdots + \frac{1}{n{\cdot}(n+1)}}_{\text{case corresponding to } n} + \frac{1}{(n+1){\cdot}(n+2)}$$
$$= \frac{n}{n+1} + \frac{1}{(n+1){\cdot}(n+2)} = \frac{n(n+2)+1}{(n+1){\cdot}(n+2)}$$
$$= \frac{(n+1)^2}{(n+1){\cdot}(n+2)} = \frac{n+1}{n+2}. \qquad \square$$

We now turn to recurrence relations. A recurrence relation (sometimes also referred to as a recursion relation) is a sequence of numbers where $a(n)$, sometimes written a_n, (the nth number) can be expressed using previous $a(k)$, for $k < n$. Examples of recurrence relations include the following:

- $a_n = 3a_{n-1} + 2a_{n-2} - a_{n-3}$. This is an example of a constant coefficient, homogeneous linear recurrence. We will examine these in more detail in a later lecture.

- $a_n = 2a_{n-1} + n^2$. This is an example of a non-homogeneous recursion. We will also examine these (in some special cases) in a later lecture.

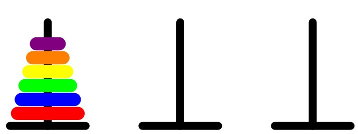
- $a_{n+1} = a_0a_n + a_1a_{n-1} + \cdots + a_na_0$. This is related to the idea of *convolution* in analysis. It shows up frequently in combinatorial problems, in particular the Catalan numbers can be found using this method.

- $a_{n,k} = a_{n-1,k-1} + a_{n-1,k}$. This is an example of a multivariable recurrence. That is we have that the number that we are interested in finding has two parameters instead of one. We have seen this particular recurrence before, namely,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

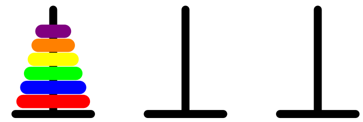
We are often interested in *solving a recurrence* which is to find an expression for $a(n)$ that is independent of the previous $a(k)$. In other words we want to find some function $f(n)$ so that $a(n) = f(n)$. In order for us to do this we need one more thing, that is *initial conditions*. These initial conditions are used to "prime the pump" and get the recursion started.

Example: The "Tower of Hanoi" puzzle relates to a problem of moving a stack of n differently sized discs on one pole to another pole, see the following picture.

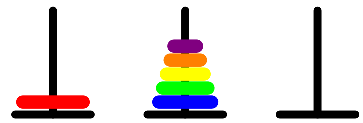


There are two rules: (1) only one disc at a time can be moved; (2) at no time can a larger disc be placed over a smaller disc. Find a recursion for T_n the minimal number of moves needed to move n discs.

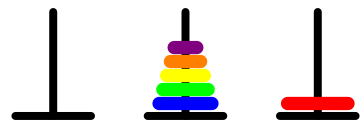
Solution: Experimenting we note that $T_1 = 1$ (i.e., move the disc and we are done), $T_2 = 3$ (i.e., move the small disc to the center pole, move the large disc to the end pole and then move the small disc back). We can keep this going, but now let us step back. As we are moving discs we must at some point move the bottom disc from the left pole to the right pole. We can use this to break the operation 0f moving all the discs into several steps as outlined below.



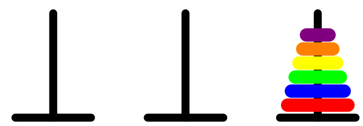
$\Downarrow$ T_{n-1} moves



$\Downarrow$ 1 move



$\Downarrow$ T_{n-1} moves



So combining we have

$$T_n = T_{n-1} + 1 + T_{n-1} = 2T_{n-1} + 1.$$

We have now found a recursion for T_n. Using this (along with the first two cases we did by hand) we have the following values for T_n.

n	1	2	3	4	5	6	7	8	9	10
T_n	1	3	7	15	31	63	127	255	511	1023

Staring at the T_n they look a little familiar. In particular they are very close to the numbers $2, 4, 8, 16, 32, \ldots$ which are the powers of 2. So we can guess that $T_n = 2^n - 1$.

This right now is a guess. To show that the guess is correct we need to do the following two things:

- Show that the initial condition(s) are satisfied.

- Show that the recurrence relationships is satisfied.

(If this reminds you of induction, you are right! Showing the initial condition(s) is satisfied establishes the base case and showing that the recurrence relationship is satisfied shows that if it is true to a certain case, then the next case is also true.)

So we now check. Our initial condition is that $T_1 = 1$, and since $2^1 - 1 = 1$ our initial condition is satisfied. Now we check that $T_n = 2^n - 1$ satisfies the recurrence $T_n = 2T_{n-1} + 1$. We have

$$2T_{n-1} + 1 = 2\big(2^{n-1} - 1\big) + 1 = 2^n - 2 + 1 = 2^n - 1 = T_n$$

showing that the recurrence relationship is satisfied. This establishes that $T_n = 2^n - 1$. This is good news since legend has it that there is a group of monks working on a copy of the Tower of Hanoi puzzle with 64 golden discs, when they are the done the world will end. We now know that it will take them

$$2^{64} - 1 = 18,446,744,073,709,551,615 \text{ moves},$$

so the world is in no danger of ending soon!

A very famous recursion is the one related to *Fibonacci numbers*. This recursion traces back to a problem about rabbit population in *Liber Abaci* one of the first mathematical textbooks about arithmetic written in 1202 by Leonardo de Pisa (also known as Fibonacci). The numbers are defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$, i.e., to find the next number add the two most recent numbers. The first few Fibonacci numbers are listed below. (Note: the book defines the Fibonacci numbers by letting $F_0 = F_1 = 1$, this is the same set of numbers but the index is shifted by 1; the definition we have given here is the more commonly accepted version.)

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55

We will study these numbers in some detail and find some "simple" expressions for calculating the nth Fibonacci number in later lectures. The Fibonacci numbers are one of the most studied numbers in mathematics (they even have their own journal dedicated to studying them). This is in part because they arise in many different interpretations.

Example: Count the number of binary sequences with no consecutive 0s.

Solution: Let $a(n)$ be the number of binary sequences of length n with no consecutive 0s. Calculating the first few cases we see the following.

n	admissible sequences	$a(n)$
0	$\emptyset$	1
1	0, 1	2
2	01, 10, 11	3
3	010, 011, 101, 110, 111	5
4	0101, 0110, 0111, 1010 1011, 1101, 1110, 1111	8

The sequence $a(n)$ looks like Fibonacci numbers, in particular it looks like $a(n) = F_{n+1}$. But we still have to prove that. Let us turn to finding a recursion for the $a(n)$.

We can group the sequences without consecutive 0s of length n into two groups. The first group will be those that have a 1 at the end and the second group will have a 0 at the end. In the first group the first $n-1$ terms is any sequence of length $n-1$ without consecutive 0s. In the second group, if the last term is 0 then we must have that the second to last term be 1 so the first $n-2$ terms is any sequence of length $n-2$ without consecutive 0s. In particular we have the following two groups

$$\underbrace{* \ * \ * \ \cdots \ * \ *}_{\substack{\text{length } n-1 \\ a(n-1) \text{ such}}} 1 \quad \text{and} \quad \underbrace{* \ * \ * \ \cdots \ *}_{\substack{\text{length } n-2 \\ a(n-2) \text{ such}}} 1\,0$$

So by the rule of addition it follows that

$$a(n) = a(n-1) + a(n-2).$$

This is the same recurrence as the Fibonacci numbers, so we have that the number of such sequences are counted by the Fibonacci numbers!

Example: A *composition* is an ordered partition, so for example the compositions of 4 are:

$$1+1+1+1, 1+1+2, 1+2+1, 2+1+1, 2+2, 1+3, 3+1, 4$$

Count the number of compositions of n with all parts odd.

Solution: Let $b(n)$ be the number of compositions of n with all parts odd. Calculating the first few cases we see the following.

n	admissible sequences	$b(n)$
1	1	1
2	$1+1$	1
3	$1+1+1,\ 3$	2
4	$1+1+1+1,\ 1+3,\ 3+1$	3
5	$1+1+1+1+1,\ 1+1+3$ $1+3+1,\ 3+1+1,\ 5$	5

Again the sequence looks like Fibonacci numbers and in particular it looks like $b(n) = F_n$. Again we still have to prove it. (After all it is possible that something looks true for the first few billion cases but eventually is false.)

We can group compositions according to the size of the last part. There are two cases, if n is odd the last part can be any of n, $n-2$, $n-4$, ..., 1. On the other hand if n is odd the last part can be any of $n-1$, $n-3$, ..., 1. In particular it is not difficult to see

$$b(n) = \begin{cases} 1 + b(n-1) + b(n-3) + \cdots & \text{if } n \text{ odd;} \\ b(n-1) + b(n-3) + \cdots & \text{if } n \text{ even.} \end{cases}$$

(The $b(n-1)$ term is when the last part is 1, the $b(n-3)$ is when the last part is 3, and so on. The 1 term for the case n odd corresponds to the composition n.)

This doesn't look like the recurrence for the Fibonacci numbers at all! But perhaps we can massage this recurrence and get the Fibonacci recurrence to pop out. In particular we note that for the case n odd we have

$$\begin{aligned} b(n) &= 1 + b(n-1) + b(n-3) + b(n-5) + \cdots \\ &= b(n-1) + \big(\underbrace{1 + b(n-3) + b(n-5) + \cdots}_{=b(n-2)}\big) \\ &= b(n-1) + b(n-2). \end{aligned}$$

For the case n even we have

$$\begin{aligned} b(n) &= b(n-1) + b(n-3) + b(n-5) + \cdots \\ &= b(n-1) + \big(\underbrace{b(n-3) + b(n-5) + \cdots}_{=b(n-2)}\big) \\ &= b(n-1) + b(n-2). \end{aligned}$$

In particular, we see that $b(n) = b(n-1) + b(n-2)$, so the sequence $b(n)$ does satisfy the same recurrence property as the Fibonacci numbers, and so we have that the $b(n)$ are the Fibonacci numbers.

Lecture 12 – April 27

Example: Find the number of compositions of n with no part equal to 1.

Solution: Let $d(n)$ denote the number of such compositions for n. Then for the first few cases we have:

n	admissible sequences	$d(n)$
1		0
2	2	1
3	3	1
4	$2+2,\ 4$	2
5	$2+3,\ 3+2,\ 5$	3
6	$2+2+2,\ 2+4,\ 3+3,\ 4+2,\ 6$	5

Looking at the numbers $0, 1, 1, 2, 3, 5$ these look familiar they look like Fibonacci numbers. (Again? Haven't we seen them enough? No, you can never see enough Fibonacci numbers!)

So now let us turn to making a recursion for $d(n)$. We can group compositions of n according to the size of the last part. Namely, the size of the last part can be $2, 3, \ldots, n-2, n$ (we cannot have the last part be size 1 since we are not allowed parts of 1, for the same reason we cannot have a part of size $n-1$ since the other part would be 1). If the last part is k then we have that the rest of the composition would be some composition of $n-k$ with no part of size 1. In particular it follows that

$$d(n) = d(n-2) + \underbrace{d(n-3) + d(n-4) + \cdots + d(2) + 1}_{=d(n-1)}$$

in particular if we group all but the first term, what we have is $d(n-1)$ and so

$$d(n) = d(n-2) + d(n-1).$$

The same recursion for the Fibonacci numbers. Since we start with Fibonacci numbers and the same recurrence is satisfied it follows that $d(n) = F_{n-1}$, i.e., the number of such compositions is counted by the Fibonacci numbers (with the index shifted by 1).

We now want to look at some important numbers that arise in combinatorics. We look at them now since they can be described using multivariable recurrences.

The *Stirling numbers of the second kind*, denoted $\left\{ {n \atop k} \right\}$, counts the number of ways to partition the set $\{1, 2, \ldots, n\}$ into k disjoint nonempty subsets (i.e., each element is in one and only one subset). For example, we have the following.

n, k	possible partitions	$\left\{ {n \atop k} \right\}$
$n{=}4$, $k{=}1$	$\{1,2,3,4\}$	1
$n{=}4$, $k{=}2$	$\{1,2,3\}\cup\{4\}$, $\{1,2,4\}\cup\{3\}$ $\{1,3,4\}\cup\{2\}$, $\{1\}\cup\{2,3,4\}$ $\{1,2\}\cup\{3,4\}$, $\{1,3\}\cup\{2,4\}$ $\{1,4\}\cup\{2,3\}$	7
$n{=}4$, $k{=}3$	$\{1\}\cup\{2\}\cup\{3,4\}$, $\{1\}\cup\{2,3\}\cup\{4\}$ $\{1\}\cup\{2,4\}\cup\{3\}$, $\{1,2\}\cup\{3\}\cup\{4\}$ $\{1,3\}\cup\{2\}\cup\{4\}$, $\{1,4\}\cup\{2\}\cup\{3\}$	6
$n{=}4$, $k{=}4$	$\{1\}\cup\{2\}\cup\{3\}\cup\{4\}$	1

The first few values of $\left\{ {n \atop k} \right\}$ are shown in the following table.

	$\left\{ {n \atop 1} \right\}$	$\left\{ {n \atop 2} \right\}$	$\left\{ {n \atop 3} \right\}$	$\left\{ {n \atop 4} \right\}$	$\left\{ {n \atop 5} \right\}$	$\left\{ {n \atop 6} \right\}$
$n=1$	1					
$n=2$	1	1				
$n=3$	1	3	1			
$n=4$	1	7	6	1		
$n=5$	1	15	25	10	1	
$n=6$	1	31	90	65	15	1

Looking at this table we see a few patterns pop out.

- $\left\{ {n \atop 1} \right\} = 1$. This is obvious since the only way to break a set into one subset is to take the whole set.
- $\left\{ {n \atop n} \right\} = 1$. This is also obvious since the only way to break a set with n elements into n subsets is to have each element of the set in a subset by itself.
- $\left\{ {n \atop 2} \right\} = 2^{n-1} - 1$. To see this note that if we break our set into two sets then it has the form of

$A \cup \overline{A}$ (i.e., A and its complement). We may assume that 1 is in A. For each remaining element it is either in A or not, the only condition is that they cannot all be in A (else $\overline{A} = \emptyset$), so there are $2^{n-1} - 1$ ways to fill in the remainder of A.

- $\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\} = \binom{n}{2}$. This can be seen by noting that there is one subset with two elements with two elements and the remaining subsets are singletons (i.e., subsets of size one). So the number of such partitions is the same as the number of ways to pick the elements in the subset with two elements which can be done in $\binom{n}{2}$ ways.

On a side note the row sums of this table are $1, 2, 5, 15, 52, \ldots$ these are also important numbers known as the Bell numbers.

We now turn to the recursion for the Stirling numbers of the second kind.

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} + k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}.$$

(Note that this is similar to the recursion for binomial coefficients. The only difference being the additional factor of k.)

To verify the recurrence we break the partitions of $\{1, 2, \ldots, n\}$ into two groups. In the first group are partitions with the set $\{n\}$ and in the second group are partitions without the set $\{n\}$. In the first group we have n in a set by itself and we need to partition the remaining $n-1$ elements into $k-1$ sets which can be done in $\left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\}$ ways. In the second group we first form k sets using the first $n-1$ elements, which can be done in $\left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\}$ ways, we now need to add n into one of the sets, since there are n sets and we can put them into any one of them the number in this group is $k\left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\}$.

The Stirling numbers of the second kind arise in many applications (even more than the Stirling numbers of the first kind). As an example consider the following.

Example: Show that the number of ways to place $n-k$ non-attacking rooks *below* the diagonal of an $n \times n$ is $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$.

Solution: Since we know that $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ counts the number of partitions of $\{1, 2, \ldots, n\}$ one method is to give a one-to-one correspondence between the rook placements and the set partitions. We will describe the positions of the chessboard as coordinates (i, j) with $i < j$. So suppose that

$$\{1, 2, \ldots, n\} = A_1 \cup A_2 \cup \cdots \cup A_k$$

is a partitioning, and suppose that

$$A_i = \{a_1, a_2, \ldots, a_{m(i)}\} \quad \text{with} \quad a_1 < a_2 < \cdots < a_{m(i)}.$$

Then place rooks at the coordinates (a_1, a_2), (a_2, a_3), …, $(a_{m(i)-1}, a_{m(i)})$. For each set we will place $|A_i| - 1$ rooks so altogether we will place

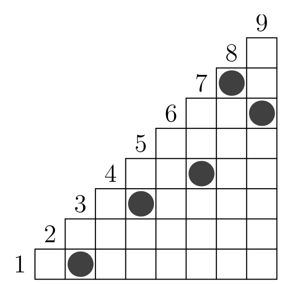
$$\sum_{i=1}^{k} \big(|A_i| - 1\big) = n - k \ \text{ rooks.}$$

Further note, that by this convention at most one rook will go in each row and in each column so that the rooks are non-attacking. Conversely, given a non-attacking rook placement we can reconstruct the partition. Namely, if there is a rook at position (p, q) then put p and q into the same subset of the partition. It is easy to see that each placement of rooks corresponds to a partition and each partition corresponds to a placement of rooks, giving us the desired correspondence.

The best way to see this is to work through a few examples. We will illustrate one here and encourage the reader to make up their own. So suppose that our partition is

$$\{1, 3, 5\} \cup \{2\} \cup \{4, 7, 8\} \cup \{6, 9\}.$$

Then the corresponding rook placement is shown below.



We also have *Stirling numbers of the first kind*, denoted $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$. These count the number of permutations in the symmetric group $\mathcal{S}_n$ that have k cycles. Equivalently, this is the number of ways to sit n people at k circular tables so that no table is empty.

If we let (abc) denote a table with persons a, b and c seated in clockwise order (note that $(abc) = (bca) = (cab)$ but $(abc) \neq (acb)$), then we have the following.

n, k	possible seatings	$\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$
n=4 k=1	(1234), (1243), (1324) (1342), (1423), (1432)	6
n=4 k=2	(1)(234), (1)(243), (2)(134) (2)(143), (3)(124), (3)(142) (4)(123), (4)(132), (12)(34) (13)(24), (14)(23)	11
n=4 k=3	(1)(2)(34), (1)(3)(24), (1)(4)(23) (12)(3)(4), (13)(2)(4), (14)(2)(3)	6
n=4 k=4	(1)(2)(3)(4)	1

The first few values of $\left[{n \atop k}\right]$ are shown in the following table.

	$\left[{n \atop 1}\right]$	$\left[{n \atop 2}\right]$	$\left[{n \atop 3}\right]$	$\left[{n \atop 4}\right]$	$\left[{n \atop 5}\right]$	$\left[{n \atop 6}\right]$
$n=1$	1					
$n=2$	1	1				
$n=3$	2	3	1			
$n=4$	6	11	6	1		
$n=5$	24	50	35	10	1	
$n=6$	120	274	225	85	15	1

Looking at this table we again see a few patterns pop out.

- $\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!$. This is obvious since everyone is sitting at one table. Fix the first person, then there are $n-1$ possible people to seat in the next seat, $n-2$ in the following seat and so on.
- $\begin{bmatrix} n \\ n \end{bmatrix} = 1$. This is obvious since the only way that this can happen is if everyone is at a private table.
- $\begin{bmatrix} n \\ n-1 \end{bmatrix} = \binom{n}{2}$. As before, one table will have a pair of people sitting and the rest will have one person each. We only need to pick the two people who will be sitting at the table together, which can be done in $\binom{n}{2}$ ways.

Looking at the the row sums of this table we have $1, 2, 6, 24, 120, 720, \ldots$, which are the factorial numbers. This is easy to see once we note that there is a one-to-one correspondence between permutations and these seatings. (Essentially the idea here is that we are refining our count by grouping permutations according to the number of cycles.)

We now turn to the recursion for the Stirling numbers of the first kind.

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

Again we note the similarity to the recursion for the binomial coefficients as well as the Stirling numbers of the second kind. Of course a small change can lead to very different behavior!

To verify the recurrence we break the seating arrangements into two groups. In the first group we consider the case where n sits at a table by themselves and in the second group n sits at a table with other people. In the first group since n is at a table by themselves this leaves $n-1$ people to fill in the other $k-1$ tables which can be done in $\left[{n-1 \atop k-1}\right]$ ways. In the second group first we seat everyone besides n at k tables, this can be done in $\left[{n-1 \atop k}\right]$ ways, now we let n sit down, since n can sit to the left of any person there are $n-1$ places that they could sit, and so the total number of arrangements in this group is $(n-1)\left[{n-1 \atop k}\right]$.

Lecture 13 – April 29

Previously we saw that if we can guess the solution to a recurrence problem then we can prove that it is correct by verifying the guess satisfies the initial conditions and satisfies the recurrence relation. But this assumes we can guess the solution which is not easy. For instance what is a way to express the Fibonacci numbers without using the recursion definition? We now turn to systematically finding solutions to some recurrences.

In this lecture we limit ourselves to solving *homogeneous constant coefficient linear recursions of order* k. This is a mouthful of adjectives, with this many assumptions we should be able to solve these recurrences! Recursions of this type have the following form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}, \qquad (*)$$

where the $c_1, c_2, \ldots, c_k$ are constants, hence the term "constant coefficient". That we only look back in the previous k terms to determine the next term explains the "order k", that we are using linear combinations of the previous k terms explains the "linear". Finally "homogeneous" tells us that there is nothing else besides the linear combination of the previous k terms, an example of something which is not homogeneous would be

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n),$$

where $f(n)$ is some nonzero function depending on n.

Our method, and even the language that we use, in solving these linear recurrences is similar to the approach in solving differential equations. So for example when we had a homogeneous constant coefficient linear differential equation of order k, we would try to find solutions of the form $y = e^{rx}$ and determine which values of r are possible. Are approach will be the same, we will try to find solutions of the form $a_n = r^n$ and determine which values of r are possible. Putting this into the equation $(*)$ we get

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}.$$

Dividing both sides of this equation by the smallest power of r (in this case r^{n-k}) this becomes

$$\begin{aligned} r^k &= c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k, \\ \text{or} \quad & r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0. \end{aligned}$$

Because the c_i's are constants this last expression does not depend on n, but is a kth degree polynomial. So we can find the roots of this polynomial (at least in theory!), and the roots are the possible values of r. So suppose that we have the roots $r_1, r_2, \ldots, r_k$ (for now we will assume that they are all distinct). We now have k different solutions and so we can combine them together in a linear combination (hence the important reason that we assumed linear). So that some solutions are of the form

$$a_n = D_1 r_1^n + D_2 r_2^n + \cdots + D_k r_k^n,$$

where $D_1, D_2, \ldots, D_k$ are constants. Now let us back to the original problem. Equation $(*)$ tells us that we look at the previous k terms to determine the next term. So in order for the recursion to get started we need to have at least k initial terms (the initial conditions). In other words we have k degrees of freedom, and conveniently enough we have k constants available. So using the k initial conditions we can solve for the k constants $D_1, D_2, \ldots, D_k$. In fact, every solution to this type of recurrence must be of this type!

We still need to determine what to do when there are repeated roots, we will address this in the next lecture.

Example: Solve the recurrence relationship $a_0 = 4$, $a_1 = 11$ and $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 2$.

Solution: First we translate the recurrence relationship into a polynomial in r. Since it is a second order recurrence this will become a quadratic, namely

$$r^2 = 2r + 3 \quad \text{or} \quad 0 = r^2 - 2r - 3 = (r-3)(r+1).$$

So that the roots are $r = -1, 3$. So that the general solutions are

$$a_n = C(-1)^n + D3^n.$$

The initial conditions translate into

$$\begin{aligned} 4 &= a_0 = C + D, \\ 11 &= a_1 = -C + 3D. \end{aligned}$$

Adding these together we have $15 = 4D$ or $D = 15/4$, substituting this into the first equation we have $C = 4 - 15/4 = 1/4$. So our solution to the recurrence is

$$a_n = \frac{1}{4}(-1)^n + \frac{15}{4}3^n.$$

Example: Recall that the Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. Find an expression for the nth Fibonacci number.

Solution: First we translate the recurrence relationship into a polynoial in r. This is again a second order recurrence so it becomes a quadratic, namely

$$r^2 = r + 1 \quad \text{or} \quad r^2 - r - 1 = 0.$$

Since it is not obvious how to factor we plug this into the quadratic formula

$$r = \frac{1 \pm \sqrt{5}}{2}.$$

So the general solution to this recursion is

$$F_n = C\left(\frac{1+\sqrt{5}}{2}\right)^n + D\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

We now plug the initial conditions to solve for C and D. We have

$$\begin{aligned} 0 = F_0 &= C + D \\ 1 = F_1 &= C\left(\frac{1+\sqrt{5}}{2}\right) + D\left(\frac{1-\sqrt{5}}{2}\right) \\ &= \frac{1}{2}(C+D) + \frac{\sqrt{5}}{2}(C-D) = \frac{\sqrt{5}}{2}(C-D) \end{aligned}$$

The first equation show that $C = -D$, putting this into the second equation we can now solve for C and D, namely $C = 1/\sqrt{5}$ and $D = -1/\sqrt{5}$. Substituting these in we have

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

This seems very unusual, the Fibonacci numbers are whole numbers and so don't have any $\sqrt{5}$ terms in them but the above expression is rife with them. So we should check our solution. One way is to plug it into the recurrence and verify that this works, but that can take some time. A quick way to check is to plug in the next term and verify that we get the correct answer. For example by the recursion we have that $F_2 = 1$, but plugging $n = 2$ in the above expression we have

$$\begin{aligned} &\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^2 \\ &\quad = \frac{1+2\sqrt{5}+5}{4\sqrt{5}} - \frac{1-2\sqrt{5}+5}{4\sqrt{5}} = \frac{4\sqrt{5}}{4\sqrt{5}} = 1. \end{aligned}$$

Notice the $\sqrt{5}$ terms all cancel out so that we are left with whole numbers.

The number that shows up in $(1+\sqrt{5})/2$ is called the golden ratio, denoted by ϕ. This number is one of the most celebrated constants in mathematics and dates back to the ancient Greeks who believed that the best looking rectangle would be one that is similar to a $1\times\phi$ rectangle. The idea being that in such a rectangle if we cut off a 1×1 square the leftover piece is similar to what we started with. However, experimental studies indicate that when asked to choose from a large group of rectangles that people tend not to go with the ones proportional to $1 \times \phi$. So perhaps the Greeks were wrong about that.

Finally, let us make one more observation about the Fibonacci numbers. We have

$$\begin{aligned}\frac{1+\sqrt{5}}{2} &= 1.618033988\ldots\\ \frac{1-\sqrt{5}}{2} &= -0.618033988\ldots\end{aligned}$$

In particular since $\left|(1-\sqrt{5})/2\right| < 1$ when we take high powers of this term we have that this becomes very small. From this we have that the second term in the above expression for F_n is smaller than $1/2$ for all values of $n \geq 0$ and so

$$F_n = \text{nearest integer to } \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

So that the rate of growth of the Fibonacci numbers is ϕ.

Example: Solve the recurrence $g_n = 2g_{n-1} - 2g_{n-2}$ with the initial conditions $g_0 = 1$ and $g_1 = 4$.

Solution: We again translate this into a polynomial in r. This becomes the quadratic

$$r^2 = 2r - 2 \quad \text{or} \quad r^2 - 2r + 2 = 0.$$

Putting this into the quadratic formula we find that the roots are

$$r = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$$

Now we have complex roots! So we might pause and think how do we change our approach. The answer is not at all, the exact same techniques work whether the roots are real or complex. So now the general solution is

$$g_n = C(1+i)^n + D(1-i)^n.$$

Putting in the initial conditions this translates into

$$\begin{aligned}1 = g_0 &= C + D.\\ 4 = g_1 &= C(1+i) + D(1-i)\\ &= (C+D) + i(C-D) \;=\; 1 + i(C-D).\end{aligned}$$

This gives $C + D = 1$ and $C - D = -3i$. Adding together and dividing by 2 we have $C = (1-3i)/2$ while taking the difference and dividing by 2 we have $D = (1+3i)/2$. So our solution is

$$g_n = \left(\frac{1-3i}{2}\right)(1+i)^n + \left(\frac{1+3i}{2}\right)(1-i)^n.$$

It should be noted that C and D are complex conjugates. This must happen in order to have the expression for g_n to be real values (which must be true by the recurrence).

Lecture 14 – May 1

As mentioned in the last lecture, solving recurrences is very similar to solving differential equations. Many of the same techniques that worked for differential equations will work for recurrences. As an example when solving the differential equation $y'' - 2y' + y = 0$ we would turn this into a polynomial $r^2 - r + 1 = 0$ and get the roots $r = 1, 1$, we would then translate this into the general solution $y = Ce^x + Dxe^x$, the extra factor of x comes from the fact that we have a double root, i.e., we could not use $Ce^x + De^x = (C+D)e^x = C'e^x$ as this does not have enough freedom for giving general solutions to the differential equation.

For recurrences we will have essentially the same thing occur. Suppose that we are working with the recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

then for finding the general solution we would look at the roots of

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0.$$

Suppose that ρ is a root of multiplicity ℓ, then the general solution has the form

$$a_n = D_1\rho^n + D_2 n\rho^n + D_3 n^2\rho^n + \cdots + D_\ell \rho^{\ell-1}\rho^n + \cdots$$

where the "$+\cdots$" corresponds to the contribution of any other roots. Notice that we have multiplicity ℓ and that we have ℓ constants, just the right amount (this should always happen!).

Example: Let $a_0 = 11$, $a_1 = 6$ and $a_n = 4a_{n-1} - 4a_{n-2}$ for $b \geq 2$. Solve the recurrence for a_n.

Solution: We first translate the recurrence into the polynomial $r^2 - 4r + 4 = 0$ which has a double root of 2. So the general solution will be of the form

$$a_n = C2^n + Dn2^n.$$

To solve for C and D we use our initial conditions. We have $a_0 = C = 11$ and $a_1 = 2C + 2D = 6$ from which we can deduce that $D = -8$. So the solution is

$$a_n = 11{\cdot}2^n - 8n2^n.$$

Example: Solve the recurrence

$$b(n) = 3b(n-1) - 3b(n-2) + b(n-3)$$

with initial conditions $b(0)$=2, $b(1)$=-1 and $b(2)$=3.

Solution: We first translate the recurrence into the polynomial

$$0 = r^3 - 3r^2 + 3r - 1 = (r-1)^3$$

which has a triple root of 1. So the general solution will be of the form

$$b(n) = C1^n + Dn1^n + En^2 1^n = C + Dn + En^2.$$

To solve for the constants we use our initial conditions. We have

$$\begin{aligned} 2 &= a_0 = C, \\ -1 &= a_1 = C + D + E, \\ 3 &= a_2 = C + 2D + 4E. \end{aligned}$$

From the first equation we have $C = 2$. Putting this in we have $D + E = -3$ and $2D + 4E = 1$. Solving these we get $D = -13/2$ and $E = 7/2$. So our solution is

$$b(n) = 2 - \frac{13}{2}n + \frac{7}{2}n^2.$$

We can occasionally take a recurrence problem which is not a constant coefficient homogeneous linear recurrence of order k and rewrite it so that it is of the form, thus allowing us to use the techniques that we have discussed so far to solve.

Example: Let $d_0 = 1$ and $nd_n = 3d_{n-1}$. Solve for d_n.

Solution: This does not have constant coefficients so our current techniques do not work. One method to solve this is to start listing the first few terms and look for a pattern (in this case it is not hard to find the pattern!). Instead let us do something that at first blush seems to make things worse, let us take the recurrence and multiply both sides by $(n-1)!$ giving

$$(n-1)!nd_n = n!d_n = 3(n-1)!d_{n-1}.$$

Now let us make a quick substitution, namely let us set $c_n = n!d_n$. Then the above recurrence becomes $c_n = 3c_{n-1}$ which has the solution $c_n = n!d_n = E3^n$, which shows that the general solution is $d_n = E3^n/n!$. Our initial condition then shows that $1 = d_0 = E$ so that our desired solution is $d_n = 3^n/n!$.

The "trick" in the last example was to find a way to substitute to rewrite the recurrence as one that we have already done. This allowed us to take a recurrence which did not have constant coefficients and treat it like one that did. This same technique can also take some recurrences which are not linear and reduce it to ones which are linear.

Example: Find the solution to the recurrence with $a_0 = 1$, $a_2 = 2$ and

$$a_n = 2\sqrt{(a_{n-1} + a_{n-2})(a_{n-1} - a_{n-2})} \quad \text{for } n \ge 2.$$

Solution: We start by rewriting this recurrence, note that for the term inside the square root it is of the form $(a+b)(a-b)$ which we can replace by $a^2 - b^2$, doing this and squaring both sides we have

$$a_n^2 = 4a_{n-1}^2 - 4a_{n-2}^2.$$

There is an obvious substitution to make, namely $b_n = a_n^2$ which relates this problem to the recurrence

$$b_n = 4b_{n-1} - 4b_{n-2},$$

with initial conditions $b_0 = a_0^2 = 1$ and $b_1 = a_1^2 = 4$. As we saw in a previous example this has the general solution

$$b_n = C2^n + Dn2^n.$$

It is easy to see that the initial conditions translate into $C = 1$ and $D = 1$. So we have

$$a_n^2 = b_n = 2^n + n2^n = (n+1)2^n.$$

Taking square roots (and seeing that we want to go with the "+" square root to match our initial conditions) we have

$$a_n = \sqrt{(n+1)2^n}.$$

Example: Solve for g_n where $g_1 = 1$, $g_2 = 2$ and

$$g_n = \frac{(g_{n-1})^2}{(g_{n-2})^{3/4}} \quad \text{for } n \ge 2.$$

Example: This is highly nonlinear. The problem is that we have division and then we also have things being raised to powers. We would like to translate this into something similar to what we have already done. Thinking back over our repertoire we recall that logarithms turn division into subtraction and also allow us to bring powers down. So let us take the logarithm of both sides. We could use any base that we want but let us go with $\log_2$ (log base 2). Then we have

$$\log_2 g_n = 2\log_2 g_{n-1} - \frac{3}{4}\log_2 g_{n-2}.$$

If we let $h_n = \log_2 g_n$ then our problem translates into

$$h_n = 2h_{n-1} - \frac{3}{4}h_{n-2}$$

with initial conditions $h_0 = \log_2 g_0 = 0$ and $h_1 = \log_2 g_1 = 1$ (our choice of base 2 was made precisely so that these initial conditions would be clean). To solve this recurrence we first translate this into the polynomial

$$0 = r^2 - 2r + \frac{3}{4} = \left(r - \frac{1}{2}\right)\left(r - \frac{3}{2}\right),$$

so that our roots are $r = 1/2, 3/2$. So the general solution for h_n is

$$h_n = C\big(\frac{1}{2}\big)^2 + D\big(\frac{3}{2}\big)^n.$$

Our initial conditions give us $0 = C + D$ and $1 = (1/2)C + (3/2)D$ or $2 = C + 3D$. It is easy to solve and find $C = -1$ and $D = 1$. So we now have

$$\log_2 g_n = h_n = -\big(\frac{1}{2}\big)^n + \big(\frac{3}{2}\big)^n = \frac{3^n - 1}{2^n}.$$

So now solving for g_n we have

$$g_n = 2^{(3^n-1)/2^n}.$$

Finally, let us look at how to deal with a recurrence relation which is non-homogeneous, i.e., a recurrence relation of the form.

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} + f(n).$$

In differential equations there are several techniques to deal with this situation. We will be interested in looking at the technique known as the "method of undetermined coefficients", or as I like to call it the "method of good guessing". This technique has definite limitations in that we need to assume that $f(n)$ is of a very specific form, namely

$$f(n) = \sum(\text{polynomial})\cdot\rho^n,$$

i.e., that $f(n)$ looks like what is possible as a homogeneous solution. (This is the same principle when doing the method of undetermined coefficients in differential equations.)

The outline of how to solve a non-homogeneous equation is as follows:

1. Solve the homogeneous part (i.e., the recurrence without the $f(n)$).
2. Solve the non-homogeneous part by setting up a solution for a_n with some coefficients to be determined by the recurrence.
3. Combine the above two part to get the general solution. Solve for the constants using the initial conditions.

Note that the order of operations is important. That is, we need to solve for the homogeneous part before we can do the non-homogeneous part and we need to solve both parts before we can use initial conditions.

The reason that we need to solve the homogenous part first is that it can influence how we solve the non-homogeneous part. So now let us look at step 2 a little more closely. So suppose that we have

$$f(n) = (j\text{th degree polynomial in } n)\rho^n.$$

We look at the homogeneous part and see if ρ is a root, i.e., part of the homogeneous solution. If it is not a root then we guess that the non-homogeneous solution will be of the form

$$a_n = (B_jn^j + B_{j-1}n^{j-1} + \cdots + B_0)\rho^n,$$

where $B_j, B_{j-1}, \cdots, B_0$ are constants which will be determined by putting this into the recursion, grouping coefficients and then making sure each coefficient is zero. (See the examples below.)

If ρ is a root then part of the above guess actually is in the homogeneous part and cannot contribute to the non-homogeneous part. In this case we need to gently nudge our solution. To do this, suppose that ρ occurs as a root m times. Then we modify our guess for the non-homogeneous solution so that it is now of the form

$$a_n = (B_jn^{m+j} + B_{j-1}n^{m+j-1} + \cdots + B_0n^m)\rho^n.$$

That is we multiply by a power of n^m to push the terms outside of the homogeneous solution.

If $f(n)$ has several parts added together we essentially do each part seperately and combine them together.

We now illustrate the approach with some examples.

Example: Solve for q_n where $q_0 = -1$, $q_1 = 1$ and

$$q_n = q_{n-2} + 3^n \quad \text{for } n \geq 2.$$

Solution: The term "$+3^n$" shows that this is a non-homogeneous recurrence, and further 3^n is of the form that we can do something with. So we first solve the homogeneous part $q_n = q_{n-2}$ which turns into the polynomial $r^2 = 1$ so that the roots are $r = \pm1$. So the homogeneous portion has the form

$$q_n = C1^n + D(-1)^n = C + D(-1)^n.$$

Now neither of the roots are 3 and so we now set up the form for the non-homogeneous part. Namely

$$q_n = E3^n,$$

we now need to determine E. To do this we substitute into the recursion. This gives

$$E3^n = E3^{n-2} + 3^n \quad \text{or} \quad \big(E - \frac{1}{9}E - 1\big)3^n = 0.$$

The second part comes from moving everything to one side and collecting the coefficients. In order for this last statement to hold (i.e., in order for it to be a solution to the non-homogeneous part) we need to have that the coefficient is 0. This means that we need $(8/9)E - 1 = 0$ so that we need to choose $E = 9/8$. So the solution for the non-homogeneous part is

$$q_n = \frac{9}{8}3^n.$$

Combining the two parts, the general solution to this recurrence is

$$q_n = C + D(-1)^n + \frac{9}{8}3^n.$$

We now take care of the initial conditions. We have

$$\begin{aligned} -1 &= q_0 &= C + D + \frac{9}{8} \\ 1 &= q+1 &= C - D + \frac{27}{8} \end{aligned}$$

Rearranging this gives

$$\begin{aligned} -\frac{17}{8} &= C + D, \\ -\frac{19}{8} &= C - D. \end{aligned}$$

Adding the two we get $2C = -36/8 = -9/2$ so that $C = -9/4$, taking the difference we get $2D = 2/8 = 1/4$ so that $D = 1/8$. So our desired solution is

$$q_n = -\frac{9}{4} + \frac{1}{8}(-1)^n + \frac{9}{8}3^n.$$

Example: Let $a_0 = 11$, $a_1 = 8$ and

$$a_n = 3a_{n-1} - 2a_{n-2} + (-1)^n + n2^n.$$

Solve for a_n.

Solution: Again we have a non-homogeneous equation, and it is of the form that we can do something with. So we first solve the homogeneous part

$$a_n = 3a_{n-1} - 2a_{n-2}.$$

This will translate into the polynomial $r^2 - 3r + 2 = 0$ which factors as $(r-2)(r-1) = 0$. So that the roots are $r = 1, 2$. So the solution to the homogeneous part is

$$a_n = C1^n + D2^n = C + D2^n.$$

We now turn to the non-homogeneous portion. Since (-1) is not a root of the homogeneous solution then that part of the solution will be of the form $E(-1)^n$. Unfortunately 2 is a root of the homogeneous solution (with multiplicity one) and so we will need to modify our guess, so instead of $(Fn+G)2^n$ we will use $(Fn^2 + Gn)2^n$. So our non-homogeneous solution has the form

$$a_n = E(-1)^n + (Fn^2 + Gn)2^n.$$

We now substitute this in and get

$$\begin{aligned} &E(-1)^n + (Fn^2 + Gn)2^n = \\ &\quad 3\big(E(-1)^{n-1} + (F(n-1)^2 + G(n-1))2^{n-1}\big) \\ &\quad - 2\big(E(-1)^{n-2} + (F(n-2)^2 + G(n-2))2^{n-2}\big) \\ &\qquad\qquad + (-1)^n + n2^n. \end{aligned}$$

The next step is to expand and collect coefficients. If we do this we get the following:

$$\begin{aligned} 0 = &\big(-E - 3E - 2E + 1\big)(-1)^n \\ &+ \big(\frac{3}{2}F - \frac{3}{2}G - 2F + G\big)2^n \\ &+ \big(-G - 3F + \frac{3}{2}G + 2F - \frac{1}{2}G + 1\big)n2^n \\ &+ \big(-F + \frac{3}{2}F - \frac{1}{2}F\big)n^2 2^n \end{aligned}$$

Each of these coefficients must be zero in order for this to be a solution. This leads us to the following system of equations (note that the last coefficient is automatically zero and so we will drop it):

$$\begin{aligned} 1 &= 6E \\ 0 &= F + G \\ 1 &= F \end{aligned}$$

This is an easy system to solve. Giving $E = 1/6$, $F = 1$ and $G = -1$ so the solution to the non-homogeneous part is

$$a_n = \frac{1}{6}(-1)^n + (n^2 - n)2^n.$$

So the general solutions is

$$a_n = C + D2^n + \frac{1}{6}(-1)^n + (n^2 - n)2^n.$$

It remains to use the initial conditions to find the constants C and D. Plugging in the initial conditions we have

$$\begin{aligned} 11 &= a_0 &= C + D + \frac{1}{6} \\ 8 &= a_1 &= C + 2D - \frac{1}{6} \end{aligned}$$

Giving $C+D = 65/6$ and $C+2D = 49/6$. Subtracting the second from the first we have that $D = -16/6 = -8/3$ and so $C = 81/6 = 27/2$. So our final solution is

$$a_n = \frac{27}{2} - \frac{8}{3}2^n + \frac{1}{6}(-1)^n + (n^2 - n)2^n.$$

(Using this formula we get $a_0 = 11$, $a_1 = 8$, $a_2 = 11$, $a_3 = 40$, $a_4 = 163$ which is what the recurrence says we should get. Wohoo!)

Lecture 15 – May 4

We can use generating functions to help solve recurrences. The idea is that we are given a recurrence for a_n, and we want to solve for a_n. This is done by letting

$$g(x) = \sum_{n \geq 0} a_n x^n,$$

we then translate the recurrence into a relationship for $g(x)$ which lets us solve for $g(x)$. We finally take the function $g(x)$ and expand it to find the coefficient for x^n.

Broken into steps we would do the following for a recurrence with initial conditions $a_0, a_1, \ldots, a_{k-1}$ and a recurrence $a_n = f(n, a_{n-1}, a_{n-2}, \ldots)$ for $n \geq k$.

1. Write $g(x) = \sum_{n\geq 0} a_n x^n$.

2. Break off the initial conditions and use the recurrence to replace the remaining terms, i.e., so we have

$$g(x) = a_0 + a_1 x + \cdots + a_{k-1}x^{k-1} + \sum_{n\geq k} f(n, a_{n-1}, a_{n-2}, \ldots)x^n.$$

3. (Hard step!) rewrite the right hand side in terms of $g(x)$ and/or other functions. Usually done by shifting sums, multiplying series, and identifying common series.

4. Now solve for $g(x)$, and then expand this into a series to read off the coefficient to get a_n. For example, this can be done using general binomial theorem or partial fractions.

The best way to see this is through lots of examples.

Example: Use a generating function to solve the recurrence $a_n{=}a_{n-1}{+}2a_{n-2}{+}1$ with $a_0{=}1$ and $a_1{=}2$.

Solution: Following the procedure for finding $g(x)$ given above we have

$$\begin{aligned} g(x) &= \sum_{n\geq 0} a_n x^n \\ &= a_0 + a_1 x + \sum_{n\geq 2} a_n x^n \\ &= 1 + 2x + \sum_{n\geq 2} \big(a_{n-1} + 2a_{n-2} + 1\big)x^n \\ &= 1{+}2x{+}\sum_{n\geq 2} a_{n-1}x^n{+}2\sum_{n\geq 2} a_{n-2}x^n{+}\sum_{n\geq 2} x^n \\ &= 1 + 2x + x\big(g(x) - 1\big) + 2x^2 g(x) + \frac{x^2}{1-x}. \end{aligned}$$

Before continuing we should see how the last step happened. We have

$$\begin{aligned} \sum_{n\geq 2} a_{n-1}x^n &= a_1x^2 + a_2x^3 + a_3x^3 + \cdots \\ &= x\big(a_1 x + a_2x^2 + a_3x^3 + \cdots\big) = x\big(g(x) - a_0\big), \end{aligned}$$

and similarly

$$\begin{aligned} \sum_{n\geq 2} a_{n-2}x^n &= a_0x^2 + a_1x^3 + a_2x^3 + \cdots \\ &= x^2\big(a_0 + a_1x + a_2x^2 + \cdots\big) = x^2 g(x). \end{aligned}$$

(This can also be done by factoring out an x^2 and then shifting the index.) Finally since $1 + x + x^2 + \cdots = 1/(1-x)$ then

$$\sum_{n\geq 2} x^n = x^2 \sum_{n\geq 2} x^{n-2} = x^2 \frac{1}{1-x}.$$

We now solve for $g(x)$, doing that we get the following.

$$(1 - x - 2x^2)g(x) = 1 + x + \frac{x^2}{1-x} = \frac{1}{1-x}.$$

Dividing we finally have the generating function

$$g(x){=}\frac{1}{(1-x)(1-x-2x^2)}{=}\frac{1}{(1-x)(1+x)(1-2x)}.$$

We now want to break this up, which we can do using the techniques of partial fractions,

$$\frac{1}{(1-x)(1+x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{C}{1-2x}.$$

Clearing denominators this becomes

$$1 = A(1{+}x)(1{-}2x){+}B(1{-}x)(1{-}2x){+}C(1{+}x)(1{-}x).$$

We can now expand and collect the coefficient of powers of x and set up a system of equations (which works great), but before we do that let us observe that this must be true for all values of x. So let us choose some "nice" values of x, namely values where most of the terms drop out. So for instance we have

$$\begin{aligned} x = 1 &\quad \text{becomes } 1 = -2A \quad \text{giving } A = -\frac{1}{2}, \\ x = -1 &\quad \text{becomes } 1 = 6B \quad \text{giving } B = \frac{1}{6}, \\ x = \frac{1}{2} &\quad \text{becomes } 1 = \tfrac{3}{4}C \quad \text{giving } C = \frac{4}{3}. \end{aligned}$$

The final ingredient will be using

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots,$$

to get

$$\begin{aligned} g(x) &= -\frac{1}{2}\cdot\frac{1}{1-x} + \frac{1}{6}\cdot\frac{1}{1+x} + \frac{4}{3}\cdot\frac{1}{1-2x} \\ &= -\frac{1}{2}\sum_{n\geq 0} x^n + \frac{1}{6}\sum_{n\geq 0}(-x)^n + \frac{4}{3}\sum_{n\geq 0}(2x)^n \\ &= \sum_{n\geq 0}\bigg(-\frac{1}{2} + \frac{1}{6}(-1)^n + \frac{4}{3}2^n\bigg)x^n. \end{aligned}$$

So we have that

$$a_n = -\frac{1}{2} + \frac{1}{6}(-1)^n + \frac{4}{3}2^n.$$

Example: Let $f(x) = \sum_{n\geq 0} F_n x^n$ where F_n are the Fibonacci numbers ($F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$). Find a simple expression for $f(x)$.

Solution: We proceed as before. We have

$$\begin{aligned} f(x) &= \sum_{n\geq 0} F_n x^n \\ &= F_0 + F_1 x + \sum_{n\geq 2} \big(F_{n-1} + F_{n-2}\big) x^n \\ &= x + x\sum_{n\geq 2} F_{n-1}x^{n-1} + x^2 \sum_{n\geq 2} F_{n-2}x^{n-2} \\ &= x + xf(x) + x^2 f(x). \end{aligned}$$

So we have $(1 - x - x^2)f(x) = x$ or

$$f(x) = \frac{x}{1 - x - x^2}.$$

We can actually use this to find an expression for the Fibonacci numbers (similar to what we got previously). To do this we will let $\phi = (1+\sqrt{5})/2$ and $\hat{\phi} = (1-\sqrt{5})/2$ then it can be checked that

$$(1 - x - x^2) = (1 - \phi x)(1 - \hat{\phi} x).$$

We now have

$$\frac{x}{1 - x - x^2} = \frac{A}{1 - \phi x} + \frac{B}{1 - \hat{\phi} x}.$$

clearing denominators we have

$$x = A(1 - \hat{\phi}x) + B(1 - \phi x) = (A + B) - (A\hat{\phi} + B\phi)x$$

We can conclude that $A = -B$ and that

$$-1 = A\hat{\phi} + B\phi = B\big(\phi - \hat{\phi}\big) = \sqrt{5}B.$$

So $B = -1/\sqrt{5}$ and $A = 1/\sqrt{5}$, giving

$$\begin{aligned} f(x) &= \frac{x}{1 - x - x^2} \\ &= \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \phi x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \hat{\phi} x} \\ &= \frac{1}{\sqrt{5}} \sum_{n\geq 0} (\phi x)^n - \frac{1}{\sqrt{5}} \sum_{n\geq 0} (\hat{\phi} x)^n \\ &= \sum_{b\geq 0} \left(\frac{1}{\sqrt{5}} \phi^n - \frac{1}{\sqrt{5}} \hat{\phi}^n \right) x^n, \end{aligned}$$

or substituting for ϕ and $\hat{\phi}$ we get

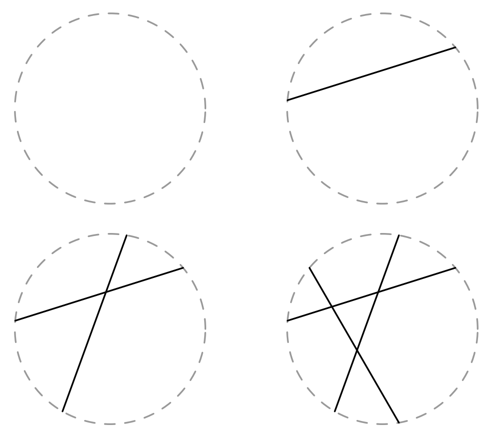
$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n,$$

as we have found previously.

(To be careful we should point out that what we have done only makes sense if our function is analytic around some value of x near 0. In our case the nearest pole for the function $f(x)$ is at $x = \hat{\phi}$ so that near $x = 0$ everything that we have done is fine. We will not worry about such matters in our course, but if you are interested in finding more about this topic read *Analytic Combinatorics* by Flajolet and Sedgewick.)

Example: Let a_n be the number of regions that n lines divide the plane into. (Assume that no two lines are parallel and no three meet at a point.) Solve for a_n using generating functions.

Solution: Let us start by looking at a few simple cases. From the pictures below we see that $a_0 = 1$, $a_1 = 2$, $a_2 = 4$ and $a_3 = 7$.



In general when we draw in the nth line we start drawing it in from ∞, every time we cross one of the $n-1$ lines we will create one additional region, and then as we head to ∞ we will create one last region. So we have the recurrence

$$a_n = a_{n-1} + n,$$

with initial condition $a_0 = 1$. (Checking we see that this gives the right answer for a_1, a_2, a_3 and a_4, which is good!)

Now we have

$$\begin{aligned} g(x) &= \sum_{n\geq 0} a_n x^n \\ &= a_0 + \sum_{n\geq 1} \big(a_{n-1} + n\big) x^n \\ &= 1 + x\sum_{n\geq 1} a_{n-1}x^{n-1} + \sum_{n\geq 1} nx^n \\ &= 1 + xg(x) + \sum_{n\geq 1} nx^n. \end{aligned}$$

The hard part about this is the second part of the sum. To handle we start with

$$\frac{1}{1-x} = \sum_{n\geq 0} x^n.$$

Taking the derivative of both sides we get

$$\frac{1}{(1-x)^2} = \sum_{n\geq 0} nx^{n-1},$$

almost what we want, we just multiply by x to get

$$\frac{x}{(1-x)^2} = \sum_{n\geq 0} nx^n.$$

(Note of course that it doesn't matter whether we start our sum at 0 or 1, the result is the same.) So we have

$$(1-x)g(x) = 1 + \frac{x}{(1-x)^2},$$

or

$$\begin{aligned} g(x) = \frac{1}{1-x} + \frac{1-(1-x)}{(1-x)^3} \\ = \frac{1}{1-x} - \frac{1}{(1-x)^2} + \frac{1}{(1-x)^3}. \end{aligned}$$

From when we started using generating functions we saw that

$$\frac{1}{(1-x)^k} = \sum_{n\geq 0} \binom{n+k-1}{n} x^n,$$

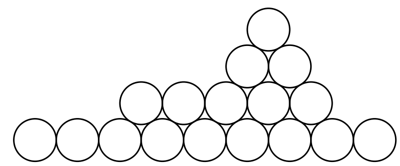
and so we have

$$\begin{aligned} g(x) &= \sum_{n\geq 0} x^n - \sum_{n\geq 0} \binom{n+1}{n} x^n + \sum_{n\geq 0} \binom{n+2}{n} x^n \\ &= \sum_{n\geq 0} \left(1-(n+1)+\frac{(n+2)(n+1)}{2}\right) x^n \\ &= \sum_{n\geq 0} \left(\frac{n^2+n+2}{2}\right) x^n. \end{aligned}$$

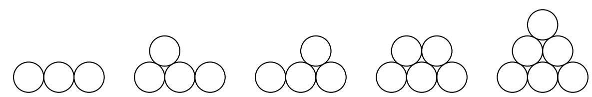
So we can conclude that $a_n = \dfrac{n^2+n+2}{2}$.

So far we have looked at examples that we could already solve using other methods. Let us try an example that we could not solve (or at least would be very hard!) with our previous methods. The following example is taken from the book *generatingfunctionology*.

Example: A block fountain of coins is a arrangement of coins in rows such that each row of coins forms a single contiguous block and each coin in the second or higher row touches two coins below (an example is shown below). Let $f(n)$ denote the number of block fountains with n coins in the bottom row. Find the generating function for $f(n)$.



Solution: We let $f(0) = 1$ (corresponding to the empty configuration). Clearly we have $f(1) = 1$ and $f(2) = 2$. The case $f(3) = 5$ is shown below.



Our next step is to find a recurrence. The fountain block either consists only of the single bottom row (one configuration) or it has another fountain block stacked on top of the bottom row. Suppose that the bottom row has length n and the second row has length j, then the second row can go in any of $n-j$ positions. Putting it altogether we have

$$f(n) = 1 + \sum_{j=1}^{n} (n-j)f(j).$$

(The 1 corresponds to the single row solution, the sum corresponds to when we have multiple rows. Note that when $j = n$ the contribution will be 0 in the sum.)

So we have

$$\begin{aligned} g(x) &= \sum_{n\geq 0} f(n)x^n \\ &= f(0) + \sum_{n\geq 1} \left(1 + \sum_{j=1}^{n}(n-j)f(j)\right) x^n \\ &= 1 + \sum_{n\geq 1} x^n + \sum_{n\geq 1} \left(\sum_{j=1}^{n}(n-j)f(j)\right) x^n \\ &= 1 + \frac{x}{1-x} + \sum_{n\geq 1} \left(\sum_{j=1}^{n}(n-j)f(j)\right) x^n. \end{aligned}$$

The difficult step is determining what to do with

$$\sum_{n\geq 1} \left(\sum_{j=1}^{n}(n-j)f(j)\right) x^n,$$

looking at it the inside reminds us of $a_0b_n + a_1b_{n-1} + \cdots + a_nb_0$, i.e., where we multiply two series together. It is not too hard to check that

$$\begin{aligned} &\sum_{n\geq 1} \left(\sum_{j=1}^{n}(n-j)f(j)\right) x^n \\ &= \big(x + 2x^2 + 3x^3 + \cdots\big)\big(f(1)x + f(2)x^2 + f(3)x^3 \cdots\big) \\ &= \frac{x}{(1-x)^2}\big(g(x) - 1\big). \end{aligned}$$

So combining we have

$$g(x) = 1 + \frac{x}{1-x} + \frac{x}{(1-x)^2}\big(g(x)-1\big),$$

or

$$\left(1 - \frac{x}{(1-x)^2}\right)g(x) = 1 + \frac{x}{1-x} - \frac{x}{(1-x)^2}.$$

Multiplying both sides by $(1-x)^2$ and simplifying we have

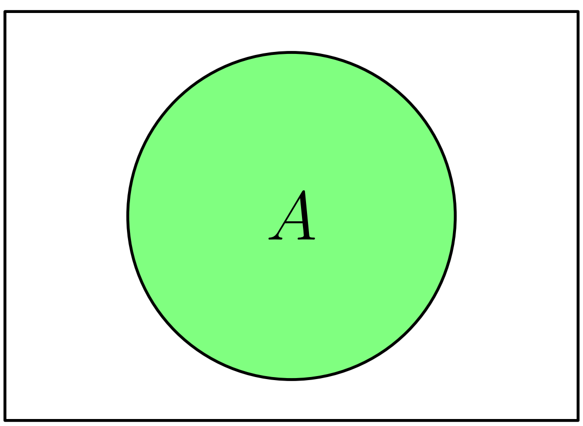
$$(1-3x+x^2)g(x) = 1-2x \quad \text{or} \quad g(x) = \frac{1-2x}{1-3x+x^2}.$$

Lecture 16 – May 6

We now return to some good old fashioned counting. We will be looking at sets. We have a collection of elements and we will let $\mathcal{U}$ denote the "univers" of all possible elements for the problem that we are counting. We will let A denote a subset of $\mathcal{U}$, $|A|$ denote the number of elements in the set A, and $\overline{A} = \mathcal{U} \setminus A$ denote the complement of A (the set of all elements in $\mathcal{U}$ not in A). Since every element is either in A or not in A we can conclude

$$|\mathcal{U}| = |A| + |\overline{A}|.$$

Pictorially we have the following picture.



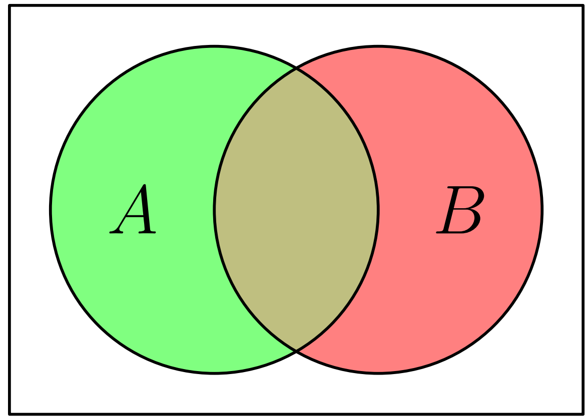
This is a simple example of a Venn diagram which gives a visual way to represent sets. We will look at Venn diagrams for 1, 2 and 3 sets, after that it gets more difficult to draw Venn diagrams (but not impossible!).

Example: There are 35 students in the class. If 15 students come to office hours then how many student's didn't come.

Solution: The set $\mathcal{U}$ is the set of all students, A the set of students who came to office hours so by the above formula we have

$$|\overline{A}| = |\mathcal{U}| - |A| = 35 - 15 = 20.$$

Not surprisingly, there is not much interesting to do with only one set. So now let us consider the situation when we have two set A and B. The Venn diagram for this situation is shown below.



Notice that this picture implies the possibility that the sets intersect. We do this because we ant Venn diagrams to be as general as possible, so the actual intersection can be empty (do not assume that sets always intersect). We let $A \cup B$ denote the union of A and B where

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\},$$

and $A \cap B$ denote the intersection of A and B where

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

de Morgan's laws

$$\begin{aligned} \overline{A \cap B} &= \overline{A} \cup \overline{B} \\ \overline{A \cup B} &= \overline{A} \cap \overline{B} \end{aligned}$$

In words de Morgan's laws says that the complement of the union is the intersection of the complements, and that the complement of the intersection is the union of the complements. To prove $\overline{A \cap B} = \overline{A} \cup \overline{B}$ we show that they have the same elements.

$$\begin{aligned} x \in \overline{A \cap B} \quad &\Leftrightarrow \quad x \notin A \cap B \\ &\Leftrightarrow \quad x \notin A \text{ or } x \notin B \\ &\Leftrightarrow \quad x \in \overline{A} \text{ or } x \in \overline{B} \\ &\Leftrightarrow \quad x \in \overline{A} \cup \overline{B} \end{aligned}$$

The other half of the law is proved similarly.

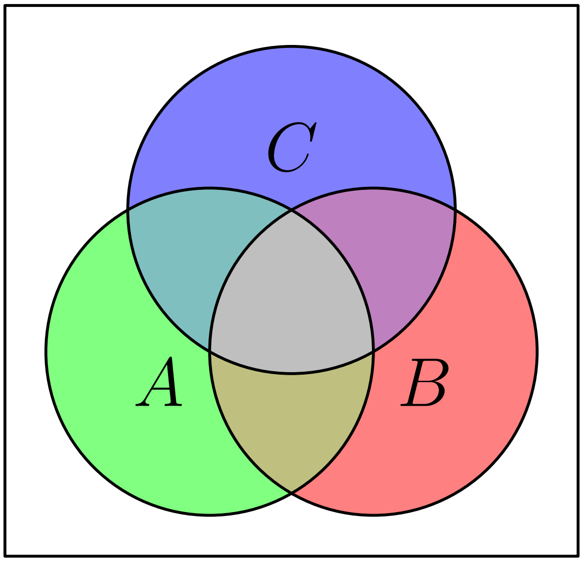
Suppose that we wanted to count the number of elements in $|A \cup B|$, a natural first guess is $|A| + |B|$. The problem is that we are double counting elements in the intersection of A and B, so we need to correct this overcounting by subtracting out. So we have

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

We can also count this by breaking $A \cup B$ into three pieces, namely $A \cap \overline{B}$ (elements in A but not in B), $A \cap B$ (elements in A and in B) and $\overline{A} \cap B$ (elements not in A but in B). So we also have

$$|A \cup B| = |A \cap \overline{B}| + |A \cap B| + \overline{A} \cap B|.$$

We now turn to the three set case. This is illustrated in the Venn diagram below.



Notice that we have split $\mathcal{U}$ into eight pieces, which is what we should expect because we are either in or not in each of the three sets. The eight pieces are

$$A \cap B \cap C,\ A \cap B \cap \overline{C},\ A \cap \overline{B} \cap C,\ A \cap \overline{B} \cap \overline{C},$$

$$\overline{A} \cap B \cap C,\ \overline{A} \cap B \cap \overline{C},\ \overline{A} \cap \overline{B} \cap C,\ \overline{A} \cap \overline{B} \cap \overline{C}.$$

The last set, by de Morgan's law is $\overline{A} \cap \overline{B} \cap \overline{C} = \overline{A \cup B \cup C}$ which are the points outside of the sets A, B and C.

Similarly as with the two set case we may want to count the union of the sets, and as before we have to be careful not to overcount. So we have

$$\begin{aligned}|A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.\end{aligned}$$

Example: At a local college there is a language program which has the three languages Russian (R), Chinese (C) and Java (J). There are 34 students attending the Russian class, 31 students taking the Chinese class, 36 students taking the Java class, 13 students taking both Russian and Chinese, 9 students taking Russian and Java and 11 students taking Chinese and Java. When this is reported to the dean they look at it and say, "Wait, I also need to know how many students are in all three!" How many students attend all three classes?

Solution: Thinking of these as sets then we are looking for $|R \cap C \cap J|$. Plugging in the information we have into the formula above we have

$$\begin{aligned}73 &= 34 + 31 + 36 - 13 - 9 - 11 + |R \cap C \cap J| \\ &= 68 + |R \cap C \cap J|.\end{aligned}$$

So $|R \cap C \cap J| = 5$, i.e., there are 5 students taking all three classes.

Example: How many n digit ternary sequences with at least one 0, one 1 and one 2?

Solution: Let us use sets to represent the situation. Let A denote the set of sequences without 0, B denote the sequences without 1 and C denote the sequences without 2. (We could alternatively let them denote the sets *with* the elements, but this is more fun.)

Phrased this way we are looking for

$$|\overline{A} \cap \overline{B} \cap \overline{C}| = |\overline{A \cup B \cup C}| = |\mathcal{U}| - |A \cup B \cup C|.$$

On the one hand we have $|A| = |B| = |C| = 2^n$, since elements in these sets are formed only using two letters. We also have $|A \cap B| = |A \cap C| = |B \cap B| = 1$ since elements in these sets are formed only using one letter (so they look like $000 \cdots 0$, $111 \cdots 1$ and $222 \cdots 2$). Finally we have $|A \cap B \cap C| = 0$ since we don't have any letters to form the sequence. Putting this altogether we have

$$|\overline{A} \cap \overline{B} \cap \overline{C}| = 3^n - \big(3 \cdot 2^n - 3 \cdot 1 + 0\big) = 3^n - 3 \cdot 2^n + 3.$$

It is always a good idea to check, so let us try 0, plugging it in we get 1, but that is not the right answer, we should have gotten 0 (there are no sequences of length 0 with at least one 0, one 1 and one 2). So where was our mistake? Looking back over what we did we see that $|A \cap B \cap C|$ will be 1 when $n = 0$ and 0 otherwise. This is because the empty string (the string with no letters) has no 0, no 1 and no 2 and so is an element in $|A \cap B \cap C|$. So if we put this back in we see that we do get the correct answer. So we have that the number of such ternary sequences is

$$\begin{cases} 3^n - 3 \cdot 2^n + 3 & \text{if } n \geq 1, \\ 0 & \text{if } n = 0. \end{cases}$$

What we have just done is a simple example of a more basic principle.

Inclusion-Exclusion Principle:
Let $A_1, A_2, \ldots, A_n \subseteq \mathcal{U}$. Then

$$\left|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}\right| = |\mathcal{U}| + \sum_{\substack{I \subseteq [n] \\ I \neq \emptyset}} (-1)^{|I|} \left| \bigcap_{j \in I} A_j \right|.$$

Before we try to prove this we first should work on trying to understand what this says. We have $[n] = \{1, 2, \ldots, n\}$, so what we are doing is summing over all nonempty subsets of $[n]$. What this is really doing is summing over all possible ways that we can intersect the sets A_i. Where we either add or subtract based on how many sets we are intersecting. For example for $n = 3$ this corresponds to the following,

$$\begin{aligned}|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |\mathcal{U}| &- \underbrace{|A_1|}_{I=\{1\}} - \underbrace{|A_2|}_{I=\{2\}} - \underbrace{|A_3|}_{I=\{3\}} \\ &+ \underbrace{|A_1 \cap A_2|}_{I=\{1,2\}} + \underbrace{|A_1 \cap A_3|}_{I=\{1,3\}} + \underbrace{|A_2 \cap A_3|}_{I=\{2,3\}} \\ &- \underbrace{|A_1 \cap A_2 \cap A_3|}_{I=\{1,2,3\}}.\end{aligned}$$

Since

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_n| &= |\mathcal{U}| - |\overline{A_1 \cup A_2 \cup \cdots \cup A_n}| \\ &= |\mathcal{U}| - |\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}|\end{aligned}$$

we automatically have the following variation of the inclusion-exclusion principle.

Inclusion-Exclusion Principle II:
Let $A_1, A_2, \ldots, A_n \subseteq \mathcal{U}$. Then

$$\big|A_1 \cup A_2 \cup \cdots \cup A_n\big| = \sum_{\substack{I \subseteq [n] \\ I \neq \emptyset}} (-1)^{|I|-1} \Big|\bigcap_{j \in I} A_j\Big|.$$

The proof of the inclusion-exclusion principle is based on a binomial identity that we encountered earlier, namely

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m \geq 1. \end{cases}$$

The way to do this is to go through every element in $\mathcal{U}$ and see how many times it gets counted on each side of the equation. We have two cases to consider.

- $x \in |\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}|$, note that x is in *none* of the sets A_i. On the left hand side this gets counted and so contributes 1. On the right hand side this gets counted when we look at the term $|\mathcal{U}|$ and since it is in none of the A_i it will not get counted any other time so the total contribution is 1.
- $x \notin |\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}|$, now x is in some of the sets A_i, let us suppose it is in m sets. On the left hand side this does not get counted and so contributes 0. On the right hand side this will get counted many times, once for the $|\mathcal{U}|$, then $\binom{m}{1}$ times for each subset it is in, then $\binom{m}{2}$ times for each pair of subsets it is in, then $\binom{m}{3}$ times for each triple of subsets it is in, and so on. In particular, the contribution on the right hand side (using signs) is
$$1 - \binom{m}{1} + \binom{m}{2} - \binom{m}{3} + \cdots = 0.$$

In both cases the left and right sides are equal for each element establishing the result.

Lecture 17 – May 8

We now look at some applications of how to use the inclusion-exclusion principle. A classic example is counting *derangement*. A derangement can be thought of as a permutation (a function $\pi : [n] \to [n]$) with no fixed point (an i so that $\pi(i) = i$). This is often rephrased as a hat-check problem, where n people go out to eat and check in their hats but when they leave the person in charge of giving back their hats has forgotten who goes with which hat and so gives them back randomly, a derangement then corresponds to the situation where no person gets their own hat back.

Example: How many derangement on n elements are there?

Solution: To use the inclusion-exclusion principle we want to first identify the sets A_i. There are many possibilities but we keep in mind two criteria. First, we want to find the intersection of the complement of the A_i. Second, we want to find sets which are easy to count when we look at their intersections. Given these two criteria we are led to

$$A_i = \{i\text{th person gets their hat}\},$$

so that $\overline{A_i}$ denotes arrangements where A_i does not get their hat back. Since no one should get their hat back we want to look at the intersection of these sets. So the number of derangements is

$$\big|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}\big| = |\mathcal{U}| + \sum_{\substack{I \subseteq [n] \\ I \neq \emptyset}} (-1)^{|I|} \Big|\bigcap_{j \in I} A_j\Big|.$$

Going through these terms we have $|\mathcal{U}|$ is the number of ways to distribute the hats which is $n!$. Now let us consider a term of the form $\cap_{j \in I} A_j$. This corresponds to arrangements that are simultaneously satisfying that elements in I get their hat back. The remaining $n - j$ can be distributed arbitrarily. So we have

$$\Big|\bigcap_{j \in I} A_j\Big| = (n - |I|)!.$$

(Importantly, we see that the only thing that is important is the number of elements of I and not which elements they are.) If we now group the sum by the size of the index set we have

$$\begin{aligned}\left|\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_n}\right| &= |\mathcal{U}| + \sum_{\substack{I\subseteq[n]\\ I\neq\emptyset}}(-1)^{|I|}\left|\bigcap_{j\in I}A_j\right| \\ &= n! + \sum_{k=1}^{n}(-1)^k\binom{n}{k}(n-k)! = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}.\end{aligned}$$

Then term $\binom{n}{k}$ shows up because there are $\binom{n}{k}$ ways to choose k out of n sets to intersect.

Note that the sum that shows up in the number of derangements looks familiar. In particular since

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} \quad\text{we have}\quad \sum_{k=0}^{\infty}\frac{(-1)^k}{k!} = e^{-1}.$$

So the number of derangements is $\approx n!/e$. In fact it can be shown that the number of derangements is the integer nearest to $n!/e$.

Using this we can conclude that the probability that if we collect hats from n people and redistribute them randomly that no one gets their hat back is $\approx 1/e$.

Example: Suppose that there are n couples that go to a theatre and sit in a row with $2n$ chairs.

(a) How many ways can the couples sit down if each person sits next to the person they came with?

(b) How many ways can the couples sit down if no person sits next to the person they came with?

Solution: For part (a) we think of pairing the couples together and then arrange the couples (which can be done in $n!$ ways). Now each couple sits down and we need to decide who sits on the left and who sits on the right which can be done in 2 ways per couple, so in total there are $2^n n!$ ways for the couples to sit down.

For part (b) we use inclusion-exclusion. If we let

$$A_i = \{\text{seatings with } i\text{th couple together}\},$$

then what we are looking for is $\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_n}$ (just right for inclusion-exclusion). The total number of ways to seat the $2n$ people is $(2n)!$. Now let us focus on a term of the from $\cap_{j\in I}A_j$. This corresponds to the situation where the couples in I are sitting together (and maybe some more as well). To count this we now think of there as being $|I|$ couples and $2n-2|I|$ singles for us to arrange, or combined $2n-|I|$ "people" to arrange. Once we are done arranging the order of the people we then again have to decide for each couple who sits on the left and who sits on the right. So altogether we have that

$$\left|\bigcap_{j\in I}A_j\right| = (2n-|I|)!2^{|I|}.$$

(Again it is important that it is the number of elements in I and not which elements in I that matters.) So as in the last example if we group by the size of the index set we have

$$\begin{aligned}\left|\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_n}\right| &= |\mathcal{U}| + \sum_{\substack{I\subseteq[n]\\ I\neq\emptyset}}(-1)^{|I|}\left|\bigcap_{j\in I}A_j\right| \\ &= (2n)! + \sum_{k=1}^{n}(-1)^k\binom{n}{k}(2n-k)!2^k \\ &= \sum_{k=0}^{n}(-2)^k\binom{n}{k}(2n-k)!.\end{aligned}$$

Again the term $\binom{n}{k}$ shows up because there are $\binom{n}{k}$ ways to choose k of the n sets to intersect. Also we can absorb the $(2n)!$ term into the sum since this corresponds to the case $k=0$.

Unfortunately, there does not seem to be any nice way to condense this formula, but at least we have a way to count it! Using a computer we see that

$$\underbrace{0}_{n=1},\underbrace{8}_{n=2},\underbrace{240}_{n=3},\underbrace{13824}_{n=4},\underbrace{1263360}_{n=5},\underbrace{168422400}_{n=6},\ldots.$$

Finally, let us look at some more variations of the inclusion-exclusion principle

> *Variation of the Inclusion-Exclusion Principle:*
> Let $A_1, A_2, \ldots, A_n \subseteq \mathcal{U}$ and let
>
> $$B_m = \{x \mid x \text{ in } \textit{exactly } m \text{ of the } A_i\}.$$
>
> Then
>
> $$|B_m| = \sum_{\substack{I\subseteq[n]\\ |I|\geq m}}(-1)^{|I|-m}\binom{|I|}{m}\left|\bigcap_{j\in I}A_j\right|.$$

The case B_0 (where we translate the empty intersection as the entire $\mathcal{U}$) gives us back the original form of the inclusion-exclusion principle. Our method to prove this is similar as before, namely to consider cases for individual elements in $\mathcal{U}$ and show that each elements contributes the same to both sides.

- x in fewer than m of the A_i. In this case $x\notin B_m$ so the contribution on the left is 0 while on the right it will not show up in any of the intersections and so its contribution is again 0.

- x in exactly m of the A_i. In this case $x\in B_m$ so the contribution on the left is 1 while on the right it will show up exactly once, namely in the intersection of the m sets it lies in. It cannot show up in any other term, so the contribution on the right is 1.

- x in $r > m$ of the A_i. In this case $x \notin B_m$ so the contribution on the left is 0. While on the right it will show up $\binom{r}{m}$ times in the intersection of m sets, $\binom{r}{m+1}$ times in the intersection of $m+1$ sets, $\binom{r}{m+2}$ times in the intersection of $m+2$ sets, ... In particular the contribution on the right hand side will be

$$\binom{r}{m}\binom{m}{m} - \binom{r}{m+1}\binom{m+1}{m} + \binom{r}{m+2}\binom{m+2}{m} - \cdots .$$

  Digging back to when we practiced identities for binomial coefficients we find the identity

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.$$

  So applying this to our situation we have

$$\binom{r}{m+i}\binom{m+i}{m} = \binom{r}{m}\binom{r-m}{i}.$$

  In particular we can rewrite our sum where we use this identity on every term and factor out the (now common) $\binom{r}{m}$ to get that the contribution on the right is

$$\binom{r}{m}\left(\binom{r-m}{0} - \binom{r-m}{1} + \binom{r-m}{2} - \cdots\right)$$

  which, by the same reasoning as last time, is 0.

Closely related to this we have the following.

Variation of the Inclusion-Exclusion Principle (II): Let $A_1, A_2, \ldots, A_n \subseteq \mathcal{U}$ and let

$$C_m = \{x \mid x \text{ in } \textit{at least } m \text{ of the } A_i\}.$$

Then

$$|C_m| = \sum_{\substack{I \subseteq [n] \\ |I| \geq m}} (-1)^{|I|-m}\binom{|I|-1}{m-1}\left|\bigcap_{j \in I} A_j\right|.$$

The case $m = 1$ corresponds to the variation that we gave in the last lecture. We will not give the proof here.

This brings us to the end of the enumerative combinatorics part of the course. Starting with the next lecture we turn to graph theory.

Lecture 18 – May 11

We now start a new area of combinatorics, namely graph theory. Graphs are at a basic level a set of objects and connections between them. As such they can be used to model all sorts of real world and mathematical objects. We start with some of the basic properties of graphs.

A *graph* G consists of two sets, a vertex set V and an edge set E, we sometimes will write this as $G = (V, E)$. The vertex set represents our objects and the edge set represents our connections between objects, we will visually denote vertices by "o". The edge set can have several different ways of describing how we connect vertices which leads to different types of graphs. The most common are:

- The set E consists of unordered pairs of vertices, i.e., $\{u, v\}$. These graphs are the most studied and are known as *undirected graphs*. Visually we represent the edges as "o—o".

- The set E consists of an ordered list of two vertices, i.e., (u, v). These graphs are known as *directed graphs*. Visually we represent the edges as "o→o".

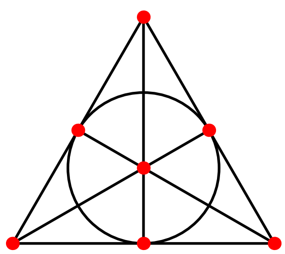
- The set E consists of subsets of vertices, usually of some fixed size, i.e., $\{v_1, v_2, \ldots, v_k\}$. These graphs are known as *hypergraphs*. These are hard to represent visually (perhaps this is one of the reasons that we do not study them as much in beginning combinatoric courses).

Usually when we think of a graph we do not think of two sets but rather as a visual drawing of points and lines connecting the points. This is great for building intuition, but can also be deceiving in that the same graph can be drawn in many different ways (we will look at this problem shortly). Another problem is that many graphs that are interesting have a *large* number of vertices (in the billions) so that if we were to draw them as points and lines the end result would be a black piece of paper. So it is good to be able to not have to rely on pictures!

An example of an undirected graph is a handshake graph which corresponds to a party with several people present and we connect an edge between two people if they shake hands. As a graph the vertices are the people and the edges correspond to handshakes. Because there is symmetry in the relationship (i.e., if I shake your hand you also shake mine) the undirected graph is the appropriate one to use. These also can be used to model social networks, in addition they are frequently used to model groups (these graphs are known as Cayley graphs).

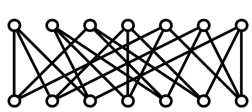
An example of a directed graph is the internet, sometimes known as the web graph. Here every vertex corresponds to a webpage and edges are hyperlinks from one webpage to another webpage. Because there does not have to be symmetry (i.e., I might link to a website about kittens but that does not mean the kitten website links to me) the directed graph is the appropriate one to use. The web graph is one of the most studied (and probably one of the most lucrative) graphs. Companies such as `Google` make a lot of money mining information from this graph.

An example of a hypergraph is the Fano plane. This is the smallest discrete geometry it consists of seven points, seven lines, where each line contains exactly three points, any two lines intersect in exactly one point and any two points are contained in one unique line. This is shown below.

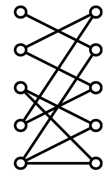


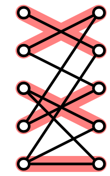
We can represent the Fano plane as a hypergraph by letting the points of the Fano plane be the vertices and the edges of the hypergraphs be the lines. So in this case every edge consists of three points.

One reason that we do not study hypergraphs is that we can model them using undirected bipartite graphs. A graph is *bipartite* if the vertices can be split into two parts $V = U \cup W$ where U and W are disjoint and such that all the edges in the graph connect a vertex in U to a vertex in W. A hypergraph $G = (V, E)$ can then be associated with a bipartite graph $H = (V \cup E, \mathcal{E})$ where $\mathcal{E}$ connects $v \in V$ to $e \in E$ if and only if the vertex v is in the edge e. So for example the hypergraph for the Fano plane can be represented using the following bipartite graph, where points are on the top lines are on the bottom. (While not as nice a picture as the Fano plane it still has all of the same information and we can still use it to determine properties of the Fano plane.)



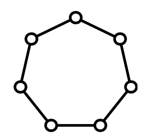
Because of their ability to model interaction between two groups, bipartite graphs are an important area of study in graph theory. As another example suppose that we want to model the social interaction between a group of five boys and five girls. There have been various relationships formed over the years and some people get along great and have fun dating while other couples don't work at all. We can model this by a bipartite graph where on the left we have the girls on the right we have the boys and we put an edge between two people if they can date. Suppose that doing this we get the graph shown below on the left.





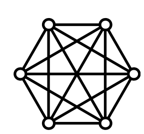
We might then ask the question, is it possible for all the people to have a date for this Friday? In this case we need to find a way to pair people up so that every person is in exactly one pairing and they get along with the person they are paired with. So looking at the graph we see that the first girl would have to be paired with the second boy, the second and fourth girls would then decide how to go with the first and third boys while the third and fifth girls would then decide how to go with the fourth and fifth boys. In particular, there are four ways that people can be paired up for dates, we show one of them above on the right.

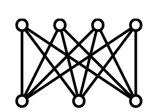
This is an example of a *matching*. A matching of a graph is a subset of edges so that each vertex is in exactly one edge. Note that this definition does not require the graph is bipartite. In the graph above there is a matching, but not all graphs have matchings. For example consider the following graph.



It is easy to see that there is no way to match up the vertices in this graph since there is an *odd* number of vertices (if there were a matching then the number of vertices would have to be a multiple of two). This graph is a special type of graph known as a *cycle*. A cycle on n vertices, denoted C_n, consists of the vertex set $\{1, 2, \ldots, n\}$ with edges $\{i, i+1\}$ for $i = 1, \ldots, n-1$ and $\{1, n\}$. If we do not include the last edge (the $\{1, n\}$) then the graph is a *path* on n vertices, denoted P_n.

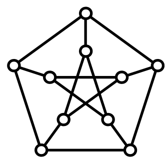
Another well studied graph is the *complete* graph K_n which consists of the vertex set $\{1, 2, \ldots, n\}$ and has all possible edges. Since edges consist of subsets of size two then there are $\binom{n}{2}$ edges in the complete graph. The graph K_6 is shown below on the left.



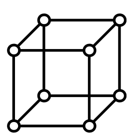


Closely related to the complete graph is the *complete bipartite* graph $K_{m,n}$ which has as its vertex set $\{v_1, \ldots, v_m, u_1, \ldots, u_n\}$ and as edges all edges of the form $\{v_i, u_j\}$. In other word we have a bipartite graph with one part of size m, one part of size n and all possible edges going between the two parts, so in total mn edges. The graph $K_{3,4}$ is shown above on the right.

One of the most famous graphs, the Fabio of graph theory if you will, is the Petersen graph. This graph consists of 10 vertices and 15 edges as shown below. This graph is a classic example/counterexample to many problems in graph theory and is a good graph to have around when testing a new idea, we will use this as an example in a later lecture.



Another important graphs are *hypercubes* Q_n (here the "hyper" refers to high dimensions, these graphs are not themselves hypergraphs). The vertex set of these graphs consist of all binary strings of length n, so that there are 2^n vertices. Two vertices are connected if the strings differ in *exactly* one entry. So for example Q_1 consists of vertices 0 and 1 with an edge connecting them, Q_2 consists of the four vertices 00, 01, 10 and 11 and form a four cycle (note that there will be no edge connecting 00 and 11 since they differ in both entries and similarly no edge connecting 01 and 10). The graph Q_3 is shown below (you should try to label the vertices with the eight binary strings of length three that would give this graph). The name hypercube comes from the idea of the vertices denoting the corners of an n-dimensional cube sitting at the origin, because binary strings are used in computers these graphs frequently arise in computer science.



Note that the hypercube is a bipartite graph. To see this we need to tell how to break up the vertices into two sets so that edges go only between these sets. This is done by letting U be the set of binary strings of length n with an *odd* number of 1s and W the set of binary strings of length n with an *even* number of 1s. Since an edge connects two vertices which differ by exactly one entry we will either increase or decrease the number of 1s by one, so that edges must go between U and W.

Let us now look at how many edges are in the hypercube. Before we do that we first need a few definitions. Two vertices u and v are *adjacent* if there is an edge connecting them, this is denoted as $u \sim v$. An edge e and a vertex v are *incident* if the vertex v is contained in the edge e (pictorially the edge connects to the vertex). The *degree* of a vertex is the number of edges incident to the vertex, equivalently the number of vertices adjacent to the vertex (assuming we do not allow loops and multiple edges), we denote the degree of a vertex by d_v or $d(v)$.

In the hypercube graph Q_n every vertex corresponds to a string of length n. To connect to this vertex we must differ in exactly one letter, since there are n possible letters then there are n different vertices that connect to the graph. In other words every vertex in the graph has degree n. If we add up all the degrees then we get $n2^n$. But adding up all the degrees we will count each edge exactly twice, so that twice the number of edges is $n2^n$ or the number of edges is $n2^{n-1}$. As an example in Q_3 there are $3 \cdot 2^2 = 12$ edges.

A graph where all of the vertices have the same degree is known as a *regular* graph. The Petersen graph is a regular graph of degree 3, the hypercube is a regular graph of degree n, the complete graph K_n is a regular graph of degree $n-1$ and the cycle C_n is regular of degree 2. Of course there are many others, regular graphs have nice properties, particularly when approaching graphs using linear algebra.

Looking at our derivation for the number of edges in Q_n we noted that by adding up the sum of the degrees that we got twice the number of edges. This is the first theorem of graph theory.

Handshake Theorem:
For a graph G let $|E|$ denote the number of edges, then
$$\sum_{v \in V} d(v) = 2|E|.$$

In particular this implies that the sum of the degrees must be an even number. So we have the following immediate corollary.

Corollary of the Handshake Theorem:
The number of vertices with odd degree must be even.

Example: Does there exist a graph on six vertices where the vertices have degree $1, 2, 3, 3, 4, 4$?

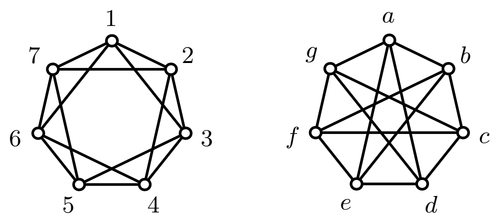
Solution: No. Summing up the degrees we get 17 which is an odd number, but by the handshake theorem if such a graph existed the sum would have to be even.

Of course, just because the sum of the degrees of a graph is even does not mean that a graph exists.

Example: Does there exist a graph on six vertices where the vertices have degree $1, 2, 3, 4, 5, 5$?

Solution: No. Summing up the degrees we get 20 so the handshake theorem does not rule it out. However if there were such a graph then it would have two vertices of degree 5, in particular these two vertices would have to be adjacent to every vertex in the graph. Or put another way, every vertex in the graph would have to be adjacent to the two vertices of degree 5, but this would imply that no vertex can have degree less than 2, in particular we could not have a degree 1 vertex as desired.

Finally, we turn to a problem mentioned earlier. Namely, we like to represent graphs visually but the way that we can draw the graph is not unique as we can place vertices where we like and draw edges as we want. For example consider the following two graphs.



These graphs have the same number of vertices (7) the same number of edges (14) all the degrees are the same, but are they two different drawings of the same graph or two different graphs altogether?

To answer this question we have to decide what we mean to say that they are the same graph, or rather they have the same structure. We say two graphs G and H are *isomorphic* ("iso"=same, "morphic"=structure) if there is a bijective (one-to-one and onto) map $\phi : V(G) \to V(H)$ so that $u{\sim}v$ in G if and only if $\phi(u){\sim}\phi(v)$ in H. Or in other words we can map the vertices of one graph to the other in a way that preserves adjacency.

In terms of the picture above the question is can we relabel the vertices of one graph and produce the other graph. Careful checking will see that if we relabel as follows $1 \mapsto a$, $2 \mapsto c$, $3 \mapsto e$, $4 \mapsto g$, $5 \mapsto b$, $6 \mapsto d$ and $7 \mapsto f$, then we preserve adjacency. In other words these two graphs are isomorphic so are two drawings of the same graph.

To show that two graphs are isomorphic then it suffices for us to find a way to relabel the vertices of one graph to produce the other graph. To find a rule for relabeling we often try to identify some vertices that must be mapped one to the other (i.e., because of degree considerations) and slowly build up the relationship.

To show that two graphs are not isomorphic we need to find some structure that one graph has that the other does not, showing they cannot be the same graph. We will pick this topic up again in the next lecture.

Lecture 19 – May 13

Our main focus in graph theory will be simple undirected graphs. Simple graphs refer to graphs without loops or multiple edges. A loop is an edge that connects a vertex to itself, i.e.,  . A multiple edge is several edges going between the same two vertices, i.e.,  . In other words we want to eliminate the possibility of having a repeated element in our sets.

A useful technique in proving statements is proof by contradiction. The underlying idea is to assume the opposite of what we are trying to prove and show that this leads to an impossible situation. This implies that our assumption is wrong, well our assumption was the opposite of what we are trying to prove, so this means what we want to prove is true. In other words we are proving something is not not true.

Example: Show that in any simple graph on two or more vertices there *must* be two vertices which have the same degree.

Solution: Suppose that it is not true, i.e., suppose that there is a graph where all vertices have different degrees. Also for convenience let us suppose that we have n vertices. Then the possible degrees in the graph are $0, 1, \ldots, n-1$. Since there are n vertices, n possible degrees and all of them are distinct this implies that we must have

- a vertex of degree 0, i.e., a vertex not connected to any other vertex; and
- a vertex of degree $n - 1$, i.e., a vertex connected to every other vertex.

But clearly we cannot have both of these vertices in the same graph. Therefore our assumption is false, and in particular there must be some two vertices with the same degree.

In the last lecture we looked at telling if two graphs were isomorphic. We saw that the way to show that two graphs were isomorphic was to produce a bijective $\phi : V(G) \to V(H)$ that preserves adjacency. But now suppose that we want to show that two graphs are *not* isomorphic. To do this we need to find some structure that one of the graphs have that the other doesn't. Some possibilities to look for.

- If the graphs are isomorphic they must have the same number of vertices and the same number of edges.
- The two graphs must have the same degree sequences (i.e., the list of degrees).
- The complement of the graphs must be isomorphic.
- Both graphs must either be connected of not connected.

- If the graphs are isomorphic then if G has a subgraph K H must also have a subgraph isomorphic to K.

These are only a few of the possibilities of things to look for. The list is nearly endless of things to look for. The nice thing is that to show two graphs are not isomorphic we only need to find a single property they don't agree on, so we usually don't have to go through the whole list of possibilities to show graphs are not isomorphic.

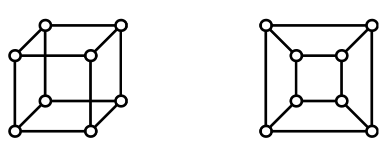
We have introduced a few terms on this list that we need to go over. The *complement* of a graph G, denoted $\overline{G}$ is the graph with the same set of vertices of G and all edges not in G. So for example the complement of $K_{m,n}$ would consist of a K_m and K_n with no edges between the two complete graphs. As another example, consider the complement of the Petersen graph, since K_{10} has degree 9 and the Petersen graph has degree 3 then the complement will be regular of degree 6 and so in particular will have 30 edges (we won't draw it). On a side note, the complement of the Petersen graph is the complement of the line graph of K_5 (but that is another story). If graphs have many edges sometimes it is easier to work with the complements which can have fewer edges.

A *subgraph* of a graph $G = (V, E)$ is a graph $H = (\hat{V}, \hat{E})$ where $\hat{V} \subseteq V$ and $\hat{E} \subseteq E$. So for example a path on n vertices is a subgraph of a a cycle on n vertices which in turn is a subgraph of K_n (in fact every graph on n vertices is a subgraph of K_n).

A graph is *connected* if between any two vertices we can get from one vertex to another vertex by going along a series of edges. That is for two vertices u, v there is a sequence of vertices $v_0, v_1, \ldots, v_k$ so that $u = v_0 \sim v_1$, $v_1 \sim v_2$, ..., $v_{k-1} \sim v_k = v$.

Another type of graph is a *planar* graph. This is a graph which can be drawn in the plane in such a way so that no two edges intersect. Of course every graph can be drawn in the plane, the requirement that edges do not intersect though adds some extra constraint.

It is important to note in this definition that a graph is planar if there is some way to draw it in the plane without edges intersecting, it does not mean that every way to draw it in the plane avoids edges intersecting. So for example in the last lecture we saw the graph Q_3 and drew it as shown below on the left. In this drawing we have edges crossing each other twice so that this is not a way to embed it in the plane without edges crossing, but we can draw it in a way without edges crossing which is shown below on the right.



To show a graph is planar we need to find a way to draw it in the plane without any edges crossing. How do we show a graph is not planar. For instance try drawing $K_{3,3}$ or K_5 in the plane, it turns out to be difficult. But is it difficult because it is impossible or is it difficult because we are not seeing some clever way to draw it. (It turns out that in this case it is difficult because it is impossible, as we will soon prove.)

The main tool for dealing with planar graphs is Euler's formula. Before we state it we first need to give one more definition. So for a drawing of a planar graph we have vertices and edges, but we also have subdivided the plane into pieces which we call faces. If we were drawing the planar graph on a piece of paper then cutting along the edges we would cut our paper into some small pieces, each piece corresponds to a face. Let V denote the set of vertices, E the set of edges and F the set of faces in the drawing of a planar graph.

Euler's formula:
For a connected planar graph

$$|V| - |E| + |F| = 2.$$

As an example in the drawing of Q_3 above we have $|V| = 8$, $|E| = 12$ and $|F| = 6$ (remember to count the "outer" face). So $8 - 12 + 6 = 2$, as the formula predicted. In the next lecture we will prove Euler's formula and use it to show that the graphs K_5 and $K_{3,3}$ are not planar as well as give some other applications.

Lecture 20 – May 15

We now prove Euler's formula. The key fact is the following. Every connected planar graph can be drawn by first starting with a single isolated vertex and then doing a series of two types of operations, namely:

1. add a new vertex and an edge connecting the new vertex to the existing graph which does not cross any existing edge; and
2. add an edge between two existing vertices which does not cross any existing edge.

One way to see this is that we can run this backwards, i.e., starting with our drawing of a planar connected graph we either delete a vertex and its incident edge if it has degree one, if no such vertex exists we remove an edge from the graph which will not disconnect the graph.

(There is a subtlety to worry about, how do we know that if all vertices have degree two or more that we can remove an edge without disconnecting the graph? To see this note that if there were a cycle in the graph (i.e., a subgraph which is a cycle, or if you prefer a sequence of edges that starts and ends at the same point) then removing an edge could not disconnect the graph since we could simply use the rest of the cycle to replace the now missing edge. Further we can grow a cycle by the following technique, we start with any vertex and move to an adjacent vertex, we then move to a new vertex other than the one we came from and continue doing this until we get to a vertex that we have previously visited. So suppose that we now have a sequence of vertices

$$v_1{\sim}v_2{\sim}v_3{\sim}\cdots{\sim}v_j$$

and that $v_j{\sim}v_i$ for some $i < j$. Then the desired cycle is

$$v_i{\sim}v_{i+1}{\sim}\cdots{\sim}v_j{\sim}v_i.$$

The key to this is since each vertex has degree two or more whenever we went into a vertex we could also exit along a different edge.)

We now proceed by induction. Our base graph consists of a single vertex. In that case we have $|V| = 1$, $|E| = 0$ and $|F| = 1$ (i.e., the outer face) and so

$$|V| - |E| + |F| = 1 - 0 + 1 = 2.$$

Now suppose that we have shown that we have $|V| - |E| + |F| = 2$ for our connected planar graph G. Suppose that we do the first type of operation as given above. Then when we are adding a new vertex and a new edge, but we do not change the number of faces (i.e., this edge does not divide a face), and so $|V'| = |V| + 1$, $|E'| = |E| + 1$ and $|F'| = |F|$ so

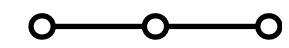
$$\begin{aligned}|V'| - |E'| + |F'| &= (|V| + 1) - (|E| + 1) + |F| \\ &= |V| - |E| + |F| = 2.\end{aligned}$$

Suppose that we do the second type of operation as given above. Then we do not add a vertex but we do add an edge, further we must split a face into two faces, and so $|V'| = |V|$, $|E'| = |E| + 1$ and $|F'| = |F| + 1$ so

$$\begin{aligned}|V'| - |E'| + |F'| &= |V| - (|E| + 1) + (|F| + 1) \\ &= |V| - |E| + |F| = 2.\end{aligned}$$

Concluding the proof.

Euler's formula is the main tool for dealing with planar graphs. In this lecture we will show how to use it to prove that some graphs are *not* planar, we will also see some additional applications in the next lecture. So let us first try to prove that the graph K_5 is not planar. Before we do this we first need to introduce the idea of counting the (bounding) edges in a face. Every face can be though of as being contained inside of some cycle and so we can count how many edges are in the cycle by starting at a vertex and walking clockwise around the face and counting how many edges we encounter before we return to our starting position, this will be the number of edges in the face. There is a small bit of subtlety to be aware of, namely consider the following graph, a planar graph on three vertices.



This graph has two edges and a single face (so as predicted by Euler's formula $3-2+1 = 2$). How many edges are in the face? Our natural instinct is to say two, but the correct answer is four. That is because as we walk around we will encounter each edge twice and we count an edge each time we encounter it. Put another way if we had cut along the edges we could not distinguish between this graph and what would have happened if we started with a four cycle.

The reason that we are using this definition is because we want to be able to count edges by using information about faces. Namely since each edge will be used in two faces (or twice in a single face) if we add up the number of edges in each face we will have double counted all the edges in the graph, i.e.,

$$2|E| = \sum_{f\in F} \left|\begin{matrix}\text{edges in}\\ \text{face } f\end{matrix}\right|.$$

We now make an observation, for every simple connected planar graph on at least three vertices, each face has at *least* three edges in it. (The reason that we limit ourselves to simple graphs is that a loop would be a face with a single edge while a pair of vertices connected by two edges would give a face bounded by two edges.) Plugging this into the above formula we have

$$2|E| = \sum_{f\in F} \left|\begin{matrix}\text{edges in}\\ \text{face } f\end{matrix}\right| \geq \sum_{f\in F} 3 = 3|F|,$$

so that $|F| \leq (2/3)|E|$. Putting this into Euler's formula we have

$$2 = |V| - |E| + |F| \leq |V| - |E| + \frac{2}{3}|E|$$

or rearranging we have

$$|E| \leq 3|V| - 6.$$

We are now ready to prove the following.

K_5 is not planar.

Suppose that K_5 were planar. Since it a simple connected graph on five vertices this would then imply by the above inequality that $|E| \leq 3|V| - 6$ but for K_5 we have $|E| = 10$ and $3|V| - 6 = 9$ so this is impossible. So K_5 is not planar.

Another graph that is difficult to draw in the plane is $K_{3,3}$, if we apply the above inequality we see that $|E| = 9$ and $3|V| - 6 = 12$. From this we can't conclude anything (just because a graph satisfies the above inequality does not automatically imply planar). We will have to work a little harder for $K_{3,3}$. One thing that we might be able to use is that the graph $K_{3,3}$ is bipartite, and bipartite graphs have nice structure. In particular we have the following.

A graph is bipartite if and only if every cycle (allowing for repetition of edges) in the graph has even length.

The assumption that we allow for edges to be repeated is not important and can be removed from the following arguments with a little care. Since it is an if and only if statement we need to prove both directions.

First let us suppose that our graph is bipartite. So we have that the vertices are in sets U and W with all edges going between U and W. Suppose that we start our cycle at a vertex in U, then after the first step we must be in a vertex in W, then after the next step we are back in U and so on. So our cycle keeps going back and forth between the two sets. After an odd number of steps we will be in W and after an even number of steps we will be in U. Since a cycle must begin and end at the same vertex our cycle must have even length.

Now let us suppose that every cycle in our graph has even length. Without loss of generality we may assume that our graph is connected (i.e., we can work on each component, a maximal connected subgraph, show that each is bipartite and so the whole graph is bipartite). We fix a vertex $\overline{v}$ and partition the vertices into V_{odd} and V_{even} by putting a vertex v into V_{odd} if there is a path of *odd* length between $\overline{v}$ and v and putting a vertex v into V_{even} if there is a path of *even* length between $\overline{v}$ and v. We claim that this is well defined, i.e., no vertex could be in both sets since if there were a path of even length and a path of odd length between $\overline{v}$ and v we could form a cycle by starting along the path of even length from $\overline{v}$ to v and then follow the path of odd length backwards from v to $\overline{v}$ (i.e., we are concatenating the paths). This new cycle would have an odd number of edges but by our assumption this is impossible. Similarly there cannot be an edge connecting two vertices v, w in V_{odd} (similarly for V_{even} since we could then form an odd cycle by going from $\overline{v}$ to v, from v to w and from w to $\overline{v}$. So this shows that we can split the vertices into two sets and all edges in the graph go between these sets, by definition this is a bipartite graph.

Since the edges bounding a face correspond to a cycle we must now be able to conclude that in a simple connected planar bipartite graph on at least three vertices each face has at least four edges, i.e.,

$$2|E| = \sum_{f\in F} \left| \begin{matrix}\text{edges in}\\ \text{face } f\end{matrix} \right| \geq \sum_{f\in F} 4 = 4|F|,$$

so that $|F| \leq (1/2)|E|$. Putting this into Euler's formula we have

$$2 = |V| - |E| + |F| \leq |V| - |E| + \frac{1}{2}|E|$$

or rearranging we have

$$|E| \leq 2|V| - 4.$$

We are now ready to prove the following.

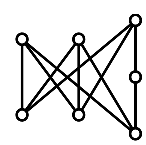
$K_{3,3}$ is not planar.

Suppose that $K_{3,3}$ were planar. Since it a simple connected bipartite graph on six vertices this would then imply by the above inequality that $|E| \leq 2|V|-4$ but for $K_{3,3}$ we have $|E| = 9$ and $2|V| - 4 = 8$ so this is impossible. So $K_{3,3}$ is not planar.

We have now shown that $K_{3,3}$ and K_5 are not planar. In some sense these are *the* problem graphs for planarity. We will make this more precise in the next lecture.

Lecture 21 – May 20

In the last lecture we saw that K_5 and $K_{3,3}$ are not planar. Clearly any graph which contains K_5 or $K_{3,3}$ as a subgraph cannot be planar (since any subgraph of a planar graph must be planar). But that is not the only possibility for ruling out planar graphs, for instance, consider the graph shown below.



This graph is found by taking $K_{3,3}$ and subdividing an edge into two edges. The resulting graph is not $K_{3,3}$ but we see that it cannot be planar, because if it were we could replace the subdivided edge by a new single edge to recover a way to draw $K_{3,3}$ in the plane. So it is not just the graph $K_{3,3}$ but rather any graph sharing similar structure (or in topology homeomorphic).

We say that a graph G contains a *topological* K_5 or $K_{3,3}$ if starting with G we can do three operations and form K_5 or $K_{3,3}$, namely:

1. Remove an edge.
2. Remove a vertex and any edge incident to the vertex.
3. Contract an edge to a single vertex. That is if $u{\sim}v$ then we remove the edge joining u and v and combine the two vertices u and v into a new vertex uv, this new vertex is adjacent to any vertex that was previously adjacent to either u or v.

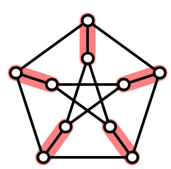
The first two operations correspond to what it takes to form a subgraph, so it is this last operation that is of the most interest to us. This is the one that allows for us to take the graph $K_{3,3}$ with a subdivided edge and contract one of the two edges to from $K_{3,3}$.

It is not hard to see that if a graph contains a topological K_5 or $K_{3,3}$ that it cannot be planar (i.e., if it were we could use it to embed K_5 or $K_{3,3}$). Amazingly this turns out to be the only situation that we need to avoid.

Kuratowski's Theorem: A graph G is planar if and only if it does not contain a topological K_5 or $K_{3,3}$.

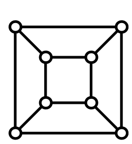
Example: Show that the Petersen graph is not planar.

Solution: To show that the Petersen graph is not planar we must find a topological K_5 or $K_{3,3}$ inside of it. Looking at the Petersen graph it is easy to see that if we contract the edges between the outer pentagon and the inner star (below highlighted in red) that the resulting graph is K_5. Therefore it contains a topological K_5 and so is not planar.



We now give one final application of Euler's formula. A *Platonic* solid is a polyhedra (equivalent of a three dimensional polygon) where each face is composed of a regular n-gon (an n sided polygon with each side and each interior angle equal) and at every corner there are exactly m faces that meet. Two examples of Platonic solids are the tetrahedron (faces are triangles and at every corner three triangles meet) and cubes (faces are squares and at every corner three squares meet). Our goal is to determine how many Platonic solids there are.

The first thing we need to do is to relate this to planar graphs. Imagine that we are holding a polyhedra in our hand that is made out of a very stretchable rubber. Then we can puncture a small hole in one of the faces of the polyhedra and start pulling it apart, stretching it out until we have flattened out the rubber. Then the corners and edges of the polyhedra form a planar graph with the corners becoming vertices and the edges becoming, well, edges. The faces of the polyhedra become the faces of the planar graph. For example, if we had started with a cube we would get the following graph.



(This connection between planar graphs and polyhedra is one reason that planar graphs are interesting to study.)

So let us now translate the conditions to be a Platonic solid into conditions on the corresponding planar graph. Each face being a regular n-gon means that each face is bounded by exactly n edges. As before we can double count edges by adding up the number of edges in each face, and so we have

$$2|E| = n|F| \quad \text{or} \quad |F| = \frac{2}{n}|E|.$$

The requirement that m faces meet at each corner tells us that at every vertex we have exactly m edges coming in. So by the handshake theorem we have

$$2|E| = m|V| \quad \text{or} \quad |V| = \frac{2}{m}|E|.$$

Putting this into Euler's formula we have

$$\frac{2}{m}|E| - |E| + \frac{2}{n}|E| = 2$$

or dividing everything by $2|E|$,

$$\frac{1}{m} + \frac{1}{n} - \frac{1}{2} = \frac{1}{|E|} > 0.$$

So in order to be a Platonic solid we need to have

$$\frac{1}{m} + \frac{1}{n} > \frac{1}{2},$$

and we must also have that $m, n \geq 3$. This only gives us five possibilities.

n	m	Platonic solid
3	3	Tetrahedron
3	4	Octahedron
3	5	Icosahedron
4	3	Cube
5	3	Dodecahedron

An octahedron has eight triangular faces, an icosahedron has twenty triangular faces and a dodecahedron has twelve pentagonal faces. Therefore the number of faces on Platonic solids are $4, 6, 8, 12, 20$. For people familiar with role playing you might notice that these numbers are on the most common type of dice that are available.

Finally, let us prove the following theorem.

Mantel's Theorem: If a simple graph on $2n$ vertices contains $n^2 + 1$ edges, then G must contain a triangle.

To contain a triangle means that there are three vertices u, v, w so that $u \sim v \sim w \sim u$ (i.e., the three vertices form a triangle). Note that this result is the best possible since the graph $K_{n,n}$ has $2n$ vertices and n^2 edges but no triangle (since a triangle would give a cycle of odd length which we know cannot happen in a bipartite graph). Also not how general this is, we are saying that no matter *how* you draw the graph if it has $2n$ vertices and n^2+1 edges it must contain a triangle (and in fact contain many triangles).

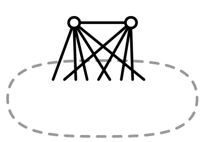
To prove this we will make use of one of the fundamental principles in combinatorics.

> *Pigeon hole principle:*
> If you distribute r balls among n bins and $r > n$ then some bin has more than one ball.

The proof is by contradiction, suppose not then the total number of balls is less than the number of bins, which by assumption is impossible. This principle seems so obvious that it is amazing that it can be used to prove anything, but in fact it is used in many beautiful proofs, including the proof we are about to give for Mantel's theorem.

We proceed by induction. For $n = 1$ it is vacuously true since there is no graph on 2 vertices with 2 edges. (This is enough to get our induction started, but suppose that we are no comfortable with this case, then for $n = 2$ it is also true since the only graph on 4 vertices with 5 edges is found by taking an edge out of a K_4 and it is easy to check that this graph has two triangles.)

Now let us suppose that we have shown that any graph on $2n$ vertices with n^2+1 edges has a triangle. Consider now a graph G on $2(n+1) = 2n+2$ vertices with $(n+1)^2+1 = n^2+2n+2$ edges. Let u and v be two vertices in G joined by an edge. Then we can draw G in the following way.



Namely as two vertices connected to an edge and then connected to the rest of the graph. Consider the graph on the vertices other than u and v, this has $2n$ vertices and so if there are n^2+1 edges in this part of the graph then by the induction hypothesis it must contain a triangle, and we are done. So we may now assume that there are $\le n^2$ edges in the part of the graph not containing u and v. Since there is 1 edge connecting u and v that says that there are $\ge 2n+1$ edges between the vertices u, v and the other $2n$ vertices. So by the pigeon hole principle there must be some vertex that is connected by two edges to the vertices u and v, i.e., there is a vertex w so that $w{\sim}u$ and $w{\sim}v$, but since we already have $u{\sim}v$ then we have our triangle and this concludes the proof.

Lecture 22 – May 22

Before we return to graph theory let us look at a nice application of the pigeon hole principle.

Let $b_1b_2\cdots b_n$ be a rearrangement of $1, 2, \ldots, n$, so for example when $n = 7$ one such rearrangement is 3521476. A subsequence of this rearrangement is $b_{i_1}b_{i_2}\cdots b_{i_k}$ where $i_1 < i_2 < \cdots < i_k$, i.e., we pick some elements of the rearrangement and still preserve their order. Some examples of subsequences in the arrangement for $n = 7$ given above include 3146, 5247, 35, 246 and so on. A subsequence is said to be increasing if $b_{i_1} < b_{i_2} < \cdots < b_{i_k}$ (for example 146 in the above) while a subsequence is said to be decreasing if $b_{i_1} > b_{i_2} > \cdots b_{i_k}$ (for example 521 in the above).

> For every rearrangement of $1, 2, \ldots, n^2+1$ there is *always* an increasing subsequence of length $n+1$ or a decreasing subsequence of length $n+1$.

This result is the best possible since if we rearrange $1, 2, \ldots, n^2$ it is possible to have no increasing or decreasing subsequence of length $n+1$. For example for $n = 9$ the (only) two sequences that do this are

$$321654987 \quad \text{and} \quad 789456123.$$

(We divide n^2 into n blocks of n then we either reverse each block and put them in order, or reverse the order of the blocks.)

Obviously we are going to use the pigeon hole principle to prove the result. The idea behind proofs which use the pigeon hole principle is to show that there is some meaningful description of the objects (bins) so that the number of objects (balls) is more than the number of ways to describe them and so two of the objects must satisfy the same description. We then use this fact to either derive a contradiction or construct a desired object.

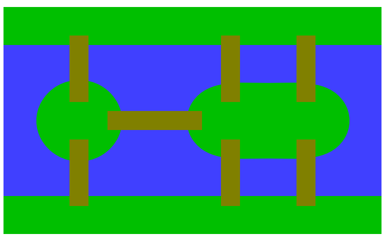
Suppose we have a rearrangement of $1, 2, \ldots, n^2+1$. Then for each $i = 1, 2, \ldots, n^2+1$ we associate a pair (I_i, D_i) where I_i is the length of the longest increasing subsequence that ends at b_i and D_i is the length of the longest decreasing subsequence that ends at b_i. so for example for the rearrangement 3521476 we get the following pairs.

$$\begin{array}{ccccccc} (1,1) & (2,1) & (1,2) & (1,3) & (2,2) & (3,1) & (3,2) \\ 3 & 5 & 2 & 1 & 4 & 7 & 6 \end{array}$$

Note that all of the pairs listed above are *distinct*. To see this suppose that $i < j$ then if $b_i < b_j$ we can take an increasing sequence ending at b_i and tack on b_j to get a *longer* increasing sequence ending at b_j showing $I_i < I_j$, similarly if $b_i > b_j$ then $D_i < D_j$. In any case we have that if $i \ne j$ then $(I_i, D_i) \ne (I_j, D_j)$.

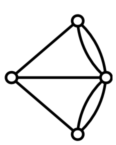
If there is no increasing and decreasing sequence of length $n+1$ or more than we must have that $1 \leq I_i, D_i \leq n$, i.e., there are at most n^2 possible (I_i, D_i). On the other hand there are n^2+1 points and so there are n^2+1 pairs (I_i, D_i), so that by the pigeon hole principle two would have to be the same. This is impossible! And so we can conclude that there must be an increasing or decreasing sequence of length $n+1$ (or more).

We now return to graph theory. We start with the first major result in graph theory given by Euler in 1736. It is based on the following story. In Königsberg (now called Kaliningrad) there was a large river that ran through the city and in the middle of the river were two large islands. Between the islands and the shore were seven bridges arranged as below.



As the story goes it was fashionable on Sunday evenings for the citizens of Königsberg to dress in their finest and walk about the city. People tried to find a route where they would return to the place they started at and cross each bridge exactly once. Try as they might people could not figure a route that would accomplish this.

Euler learned of the problem and he approached it by first translating it into a question about graph theory. He let each piece of land be a vertex and each bridge an edge connecting two vertices. This gives us the following multi-graph (a multi-graph because it has two edges between two vertices).



Translating this into graph theory we are looking for a cycle in the graph (i.e., we start at a vertex and move along edges and end at the vertex we started at) which visits each vertex at least once and uses each edge *exactly* once (the main condition is that we are using each edge exactly once). Such a cycle is called an *Euler cycle*. Euler made two observations, first if the graph is to have an Euler cycle then it must be connected because we can get from any vertex to any other vertex by using the cycle. The second observation is that the degree of each vertex must be even. To see this if we look at each vertex we will at some point come in and then we will leave via a different edge, the idea here is that we can pair up the edge we came in with the edge that we came out and we do this every time we return to the vertex.

Now looking at the graph above it is easy to see that it is impossible to have an Euler cycle because the degrees of the graph are $3, 3, 3, 5$, not even one even degree!

So we now know when a graph does not have an Euler cycle (i.e., it is not connected or has odd degree vertices), but when does a graph have an Euler cycle? Amazingly the necessary conditions are also sufficient conditions.

> A graph has an Euler cycle if and only if it is connected and every vertex has even degree.

We have already seen that if a graph has an Euler cycle that it is connected and every vertex has even degree. So we need to go in the opposite direction, i.e., we need to show that if the graph is connected and every vertex has even degree that it has an Euler cycle. We do this by giving a construction that will make an Euler cycle and show that the requirements for the construction are precisely those that the graph is connected and every vertex has even degree.

We start with our graph and pick a vertex v. We start by forming a cycle by going out along an edge and then once we get to a new vertex we go along a previously unused edge. (The important part is that we will always use a previously unused edge.) We keep doing this until we get to a vertex that has no previously unused edge. Since every vertex has even degree every time we come into a vertex we can exit along a different edge (i.e., we can pair edges up) the only vertex where this is not true is the vertex that we started at and so we now have a cycle where each edge is used at most once.

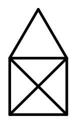
If we have used all of the edges then we are done. So suppose that we have not used all of the edges. Since the graph is connected there must be some vertex w in the cycle that is incident to a previously unused edge. We start the process again at this vertex and create another cycle that uses each edge at most once. We now glue the two cycles together, i.e., go from v to w along part of the first cycle, use the second cycle to get from w to w and then finally use the remaining part of the first cycle to get from w to v. Thus we have a longer cycle where each edge is used at most once. We continue this process until we create a cycle which uses each edge exactly once, our desired Euler cycle.

Related to Euler cycles are Euler paths, the only difference being that we do not return to the vertex that we started at (i.e., we form a path, not a cycle). It is easy to see that we have the following condition.

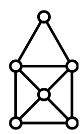
> A graph has an Euler path if and only if it is connected and exactly two vertices have odd degree.

Note that the Euler path must start and end at the vertices with odd degree.

Example: Can the following figure be drawn without lifting a pencil or crossing each edge twice?



Solution: Making this a graph by putting a vertex where any edges come together we get the following.



In particular we see that the degrees of this graph are $2, 3, 3, 4, 4, 4$. Since it has exactly two vertices with odd degree this graph has an Euler path which would correspond to a drawing of the graph where we draw each edge exactly once and we never lift our pencil off the paper. We can even see that if we wanted to be able to draw it we must start at either the lower left or the lower right corner (i.e., the vertices with odd degree.

Lecture 23 – May 27

There is a directed analog for Eulerian cycles. Recall that in a directed graph each edge has an orientation which we marked with an arrow. So an Eulerian cycle for a directed graph is one that uses each edge exactly once, visits each vertex and also on each edge goes in the direction indicated by the orientation. For the undirected graphs we used the fact that every time we came in we also had to leave so that we paired up edges and so the degree at each vertex had to be even. For directed graphs we have the same situation except now we have that when we pair edges up we pair an edge going *in to* the vertex with one going *away from* the vertex. This gives us the directed analog for Eulerian cycles.

> A directed graph has an Euler cycle if and only if it is connected and at every vertex the number of edges directed into the vertex equals the number of edges directed away from the vertex.

We can use the directed Euler cycles to construct *de Bruijn* sequences. A de Bruijn sequence is a way to pack all 2^n binary words of length n into a single binary word of length 2^n where each binary word of length n occurs exactly once as n consecutive digits (where we also allow for wrapping around). For example for $n = 1$ we have the binary words 0 and 1 so a de Bruijn sequence in this case is 01. For $n = 2$ we have the binary words 00, 01, 10 and 11 and it can be checked that 0011 is a de Bruijn sequence. For $n = 8$ there are two possible de Bruijn sequences 00011101 and 00010111, below we check that the first one is a de Bruijn sequence (remember that we allow for wrap around!).

word	location
000	00011101
001	00011101
010	00011101
011	00011101
100	00011101
101	00011101
110	00011101
111	00011101

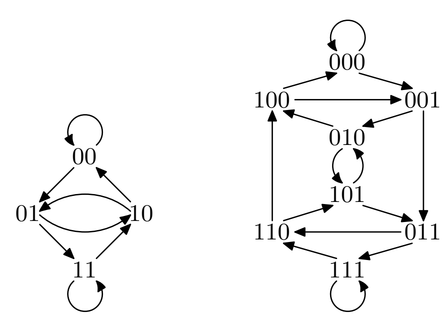
The first question that should be asked is do such sequences always exist, and if so how to construct them. They always exist, and the method to construct them involves directed Eulerian cycles (which conveniently we just discussed!).

To make a de Bruijn sequence for binary words of length n, we define a directed graph $\mathcal{D}_n$. The vertices of $\mathcal{D}_n$ are binary words of length $n-1$ (so that there are 2^{n-1} vertices) and we have directed edges going from

$$a_1a_2a_3\ldots a_{n-1} \to a_2a_3\ldots a_{n-1}a_n.$$

In other words we put a directed edge if the last $n-2$ digits of the first vertex matches the first $n-2$ digits of the second vertex. (Note in particular that each *edge* is associated with a binary word on n letters and each binary word on n letters is associated with a unique edge. So de Bruijn sequences then say that we need to use all of the edges.)

The graphs $\mathcal{D}_3$ and $\mathcal{D}_4$ are shown below.



We now make some observations about this graph.

- The graph $\mathcal{D}_n$ is connected. To see this we have to describe a way to get from any vertex to any other vertex. Suppose that we have vertices $a_1a_2a_3\ldots a_{n-2}a_{n-1}$ and $b_1b_2b_3\ldots b_{n-2}b_{n-1}$, then we can use the following edges to connect the vertices:

$$\begin{array}{rcl} a_1a_2a_3\dots a_{n-2}a_{n-1} & \to & a_2a_3\dots a_{n-2}a_{n-1}b_1 \\ a_2a_3\dots a_{n-2}a_{n-1}b_1 & \to & a_3\dots a_{n-2}a_{n-1}b_1b_2 \\ \cdots & \to & \cdots \\ a_{n-1}b_1b_2b_3\dots b_{n-2} & \to & b_1b_2b_3\dots b_{n-2}b_{n-1} \end{array}$$

Basically at each step we take off one term from the first vertex and add on a term from the second vertex. So after $n-1$ steps we can get from any vertex to any other vertex so the graph is connected.

- The number of edges going into a vertex is equal to the number of edges going out of a vertex. More precisely there are 2 edges coming in and 2 edges going out. For a vertex $a_1a_2\dots a_{n-2}a_{n-1}$ the edges coming in are

$$\begin{array}{rcl} 0a_1a_2\dots a_{n-2} & \to & a_1a_2\dots a_{n-2}a_{n-1} \\ 1a_1a_2\dots a_{n-2} & \to & a_1a_2\dots a_{n-2}a_{n-1} \end{array}$$

while the edges going out are

$$\begin{array}{rcl} a_1a_2\dots a_{n-2}a_{n-1} & \to & a_2\dots a_{n-2}a_{n-1}0 \\ a_1a_2\dots a_{n-2}a_{n-1} & \to & a_2\dots a_{n-2}a_{n-1}1 \end{array}$$

By the above two items we see that the graph $\mathcal{D}_n$ has an Euler cycle (in fact it can be shown that it has $2^{2^{n-1}-n}$ Euler cycles). The last thing to note is that every Euler cycle corresponds to a de Bruijn sequence (and vice-versa). For example in the graph $\mathcal{D}_3$ it is easy to find an Euler cycle

$$00 \to 00 \to 01 \to 11 \to 11 \to 10 \to 01 \to 10 \to 00.$$

To turn it into a de Bruijn sequence we take each edge and record the letter that it adds onto the end of the word. Doing that for the above sequence we get 01110100 which is a de Bruijn sequence (note that this is the same as the de Bruijn sequence listed earlier where we have shifted the entries, but since we are dealing with an Euler cycle shifting the entries only corresponds to changing the starting location of our Euler cycle, it is still the same cycle. The way to see that this works is that every edge is coded by a word of length n, namely the first $n-1$ digits tells us the location of the originating vertex and the last $n-1$ digits tells us the location of the terminal vertex. Looking at a term and the previous $n-1$ terms (where we cycle back) tells us the edge we are currently on. Since we use each edge exactly once each binary word of length n must show up exactly once.

de Bruijn sequences have an interesting application as part of a magic trick (I learned this from Ron Graham and Persi Diaconis who are writing a book about magic and mathematics). Namely a magician is performing a magic show and he throws a deck of cards out into the back of the audience. The person who catches it is told to cut the deck (which is to split the deck in half and then put the top half on the bottom) then hand it to the person to their right, this is repeated until five people have cut the deck. (This is to show that their is no one in cahoots with the magician, i.e., the magician now has no idea what card is on top.) He now has the fifth person take off the top card, and hand it back to the fourth person who again takes off the top card and hands it back to the third person and so on until the last person.

He now looks at the five people holding cards and he tells them he is going to read their minds and tell them which card they have and asks them to each focus on the card that they have. He stares intensely for a fes second and then declares "I am having a little trouble receiving your mental signals, I have discovered that sometimes the signals from the red cards overpower the black cards, could everyone holding a red card please sit down." At this point anyone holding a red card sits down. He continues "Ah, much better. I now can see the cards clearly and you have a ..." at which point he lists everyones card.

So what happened. Well one possibility is that the magician can actually read minds, but there is an easier explanation. Noticed that he asked for people to sit down, this was a way to send a signal to him about the order of the black and red cards. The magician actually planned ahead and did not toss out a random deck but instead had a deck where he had arranged 32 cards (note that $2^5 = 32$) in a specific order so that any five consecutive cards had a unique pattern of red and black cards (this is easily done using a de Bruijn sequence for $n = 5$). The fact that he had them cut cards did not change the de Bruijn sequence, the only important thing was that he had five consecutive cards and he was able to determine the order of the red and black cards. Once he knew this he knew where he was in the de Bruijn sequence and he could then say which cards they were by memorizing the locations (or having a cheat sheet to look off).

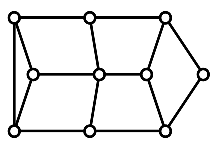
The idea behind Euler cycles is that we have a cycle in the graph which uses each *edge* exactly once. We can ask a similar question about a cycle which uses each *vertex* exactly once. Such a cycle is called a *Hamilton cycle* after a mathematician who studied these cycles (and also invented a related game about trying to find a way to visit twenty different cities so that each city is visited exactly once).

While it is easy to characterize graphs which have Euler cycles, it is difficult to characterize graphs which have Hamilton cycles (also known as hamiltonian graphs). There are some clearly necessary conditions, i.e., the graph must be connected and it cannot have any bridges), on the other hand there are also some sufficient conditions. But there are no set of conditions which are necessary and sufficient that work for all graphs. An example of a sufficient condition is Dirac's Theorem (we will not prove it but its proof can be found in most graph theory textbooks).

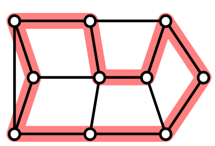
> *Dirac's Theorem*
> If G is a graph on $n \geq 3$ vertices and the degree of each vertex is $\geq n/2$ then G is hamiltonian.

In general to show a graph is hamiltonian we only need to find a cycle that uses each vertex exactly once. To show a graph is not hamiltonian we need to show that it is impossible to find a cycle that uses each vertex exactly once (in this case it pays to be clever, i.e., to avoid looking at every possible cycle).

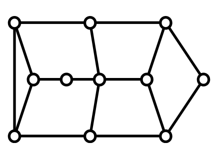
Example: Show that the graph below is hamiltonian.



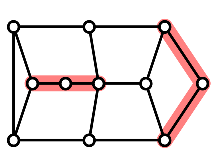
Solution: We only need to find a hamiltonian cycle. This is not too hard and one possibility is shown below.



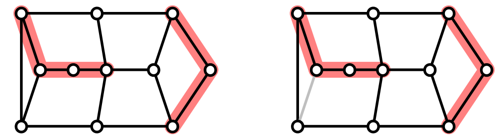
Example: Show that the following graph does not have a hamiltonian cycle.



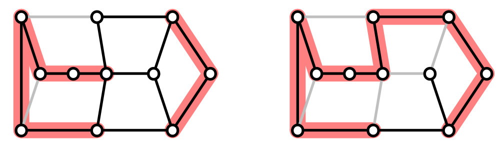
Solution: Note that this is almost the same graph as before, the only difference is that now we have taken an edge and split it into two edges, but apparently even this small change is enough to force us to not be hamiltonian. Let us make an observation, if the graph has a hamiltonian path then any vertex with degree two must have both edges in the cycle. So looking at our graph any hamiltonian cycle must have the following four edges.



By symmetry we can assume that for the vertex in the middle on the left it connects to the vertex in the top on the left, as shown below on the left.



Once two edges incident to a vertex are used in a hamiltonian path any remaining incident edges to that vertex cannot be used so we can remove it from further consideration (as shown above on the right). Using this observation we get the following graphs.



Looking at the graph on the right it is easy to see that no matter which of the remaining edges we use we will run into a problem. In particular we cannot close up and form a hamiltonian cycle. (We can make a path that uses all the vertices exactly once, such a path is called a hamiltonian path.)

Lecture 24 – May 29

A special case of hamiltonian cycles are hamiltonian cycles on the n-dimensional hypercube Q_n. These are known as Gray codes (named after Frank Gray, an engineer at Bell Labs). Because of their connection to binary words they are studied in computer science. They also are useful if we want to look through all possible combination of using or not using some collection of objects since the edges in the hypercube only change in one digit so this corresponds to changing the state of just one object at each step.

We now show how to construct a hamiltonian cycle for Q_n. First note that this holds for $n = 1$ since we have $0 \to 1 \to 0$ (in this case we will allow ourselves to go back over the edge we used). For $n = 2$ we have essentially only one cycle (that is we can assume that we start at $00\cdots 0$ and that we pick some direction), namely

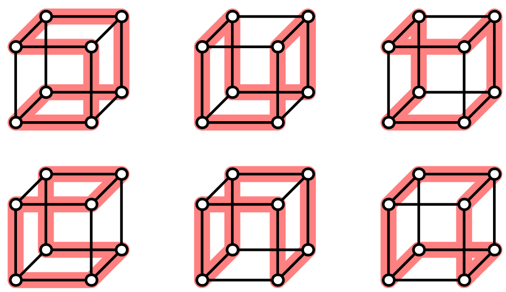
$$00 \to 01 \to 11 \to 10 \to 00.$$

In general if we have a hamiltonian cycle for Q_n, call it R, then to construct a hamiltonian cycle for Q_n we look at $0R$ and $1\overline{R}$, i.e., we put a 0 in front of everything in R (the $0R$) and then we put a 1 in front of everything in R where we run in reverse order (the $1\overline{R}$). For example, using the above Gray code for $n = 2$ we get the following Gray code for $n = 3$

$$\underbrace{000\to001\to011\to010}_{0R} \to \underbrace{110\to111\to101\to100}_{1\overline{R}} \to 000.$$

It is easy to see that this will visit all vertices and that each two consecutive entries are adjacent and so we have a desired hamiltonian cycle.

Of course this is not the only way to form a Gray code. In general there are many ways that we can go about forming a Gray code and we can choose different Gray codes which have different desired properties. One natural question to ask is how many are there, for $n = 3$ there are essentially 6 different Gray codes and these are shown below.

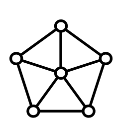


For $n = 4$ there are $1,344$ different Gray codes and for $n = 5$ there are $906,545,760$ different Gray codes (we will not draw them here). For $n = 6$ only an approximate number is known.

We now turn to coloring graphs. There are essentially two things that can be colored in a graph, vertices and edges. We will restrict ourselves mostly to the case of coloring the vertices (note that coloring edges is usually equivalent to coloring the vertices of the line graph). So a coloring is an assignment of colors, usually listed as $1, 2, \ldots, k$, to the vertices. Equivalently, a coloring is a function $\phi : V \to \{1, 2, \ldots, k\}$.

Our definition of coloring is very general so most of the time we will restrict our colorings so that they are *proper colorings*, which are colorings of a graph where no two adjacent vertices have the same color. The minimum number of colors (i.e., smallest value of k) needed to have a proper coloring for a graph G is called the *chromatic number* of the graph and is denoted $\chi(G)$.

Example: Find the chromatic number for K_n, bipartite graphs, C_n and the graph shown below.



Solution: For the complete graph since every vertex is adjacent to every other vertex they must all have different colors and so $\chi(K_n) = n$. We know a graph is bipartite if we can split the vertices into two sets U and W so that all edges go between U and W, in particular if we color all the vertices in U blue and all the vertices in W red then we have a proper coloring, so a bipartite graph only needs two colors. This gives another characterization of bipartite graphs.

A graph G is bipartite if and only if $\chi(G) = 2$.

For the graph C_n we note that if n is even that we have a bipartite graph and so we only need two colors. If n is odd then the graph is not bipartite so we need more than two colors, it is not hard to see that we can use only three, namely we go around and alternate red–blue–red–blue–... and then when we get to the last vertex we see that it can neither be red or blue so we color it using a third color. In particular we have

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

Finally for the last graph we note that it contains the graph C_5 on the outer edge. Just to color those vertices we need at least three colors, then finally the central vertex is adjacent to all the other vertices so it must be a different color then those used on the outer cycle so we need at least four colors, and it is not hard to color the vertices of the graph using four colors so the chromatic number is 4.

The chromatic number has some interesting applications. If we look at a graph with a proper coloring then grouping the vertices together with the same color (known as the color classes) we see that there are no edges between red vertices, no edges between blue vertices, and so on. In graph terminology that says that the vertices in each color class form an independent set (a set where no two vertices are adjacent). So finding the chromatic number of a graph is the same as finding the minimum number of sets that we can group vertices into so in each set no two vertices are adjacent.

For example, suppose that we want to set up a schedule for committee meetings. We want to have as few meeting times as possible but there are some people who serve on more than one committee. In the latter case any two committees with more than one person serving on them cannot meet at the same time. Let us represent our situation using a graph where each vertex is a committee and we draw an edge between any two committees that share a member (i.e., committees that cannot meet simultaneously). Then the chromatic number of the graph is the *minimal* number of committee meeting times needed so that each committee can meet once. Further, each color class tells us what committees should be meeting simultaneously.

As another example, suppose that we want to store chemicals in warehouses. Some chemicals interact and so must be stored in separate warehouses while other chemicals do not and can safely be stored together. Again let us represent the situation using a graph where each vertex is a chemical and we draw an edge between two chemicals if they interact. Then the chromatic number is the *minimal* number of warehouses needed to safely store all the chemicals. Further, each color class tells us what chemicals should be stored together to achieve the minimum.

Given that the chromatic number is a useful property of a graph we would like to have a way to find it. This is a nontrivial problem in general, but there are some easy estimates (though generally speaking these are quite poor).

> Let $\Delta(G)$ be the maximum degree of a vertex in a graph G. Then $\chi(G) \le \Delta(G) + 1$.

The proof of this is very easy. We start coloring and we color greedily in that at every vertex we use the smallest color not used on any vertex adjacent to the current vertex. Since each vertex is adjacent to at most $\Delta(G)$ other vertices, i.e., so it is adjacent to at most $\Delta(G) + 1$ other colors, we only need to use $\Delta(G) + 1$ colors to make a proper coloring of the graph. This Bound is far from best possible and it can be shown that the only graphs for which $\chi(G) = \Delta(G) + 1$ are K_n and odd cycles (C_n with n odd).

To get a lower bound we first make the following observation.

> If H is a subgraph of G then $\chi(H) \le \chi(G)$.

This is easy to see since any proper coloring of G automatically gives a proper coloring of H. So H would never need more colors than it takes to color G (and depending on the graph can sometimes use far fewer). Clearly one type of graph which has a high chromatic number are complete graphs, so one way to look for a lower bound for the chromatic number is to look for large complete graphs inside the graph we are trying to color. This gives us the following result.

> Let $\omega(G)$ denote the clique number of the graph G, i.e., the size of the largest complete graph that is a subgraph of G. Then $\chi(G) \ge \omega(G)$.

Looking at the last result it is tempting to think that the chromatic number of a graph is somehow a "local" property. In other words the main force in giving us a high chromatic number is to have a large clique. This is not the case, and in fact the opposite counterintuitive fact is true. Namely, that there are graphs which have high chromatic number but "locally" look like trees (which are simple bipartite graphs we will talk about in the next lecture). More precisely, let the girth of a graph to be the length of the smallest cycle without repeated edges.

> *Theorem (Erdős):*
> For any k there exists graphs with $\chi(G) \ge k$ and girth at least k.

The most famous problem in graph theory (which became the driving force of graph theory for more than a century) is the four color problem. This dates back to the 1850's when someone noticed that when coloring the counties of England that they only needed four colors so that no two adjacent counties had the same color. (Adjacent meant that they shared some positive length of distance. For instance in the United States the four corners is a point where four states meet at a point, namely Utah, Colorado, New Mexico and Arizona we would not consider Arizona and Colorado adjacent. We also make the basic assumption that counties are contiguous.) The question then became is this always possible for any map.

We can relate this to graph theory by putting a vertex for every county (or state, country, whatever) and connect two vertices when two counties share a border. The resulting graph is planar, and so the question became can every planar graph be colored with four or fewer colors?

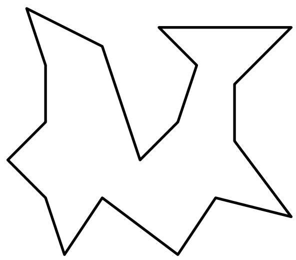
Since every planar graph has a vertex of degree at most 5 it is not hard to show by induction that the chromatic number for a planar graph is at most 6. In the late 1800's Kempe found a "proof" that the chromatic number of the planar graph was 4 but it was later shown to have a defect and was modified to show that the chromatic number was at most 5. The final proof was finally settled in the 1970's by Appel and Haken.

> *Theorem (Appel and Haken):*
> If G is a planar graph then $\chi(G) \le 4$.

This was one of the most controversial proofs in mathematics because it heavily relied on computers and could not be verified by hand. A shorter proof (that still relied on computers) was subsequently found and most people generally accept the four color theorem as true.

Interestingly enough there are similar statements for other surfaces. For instance any graph drawn on the surface of the torus with no edges crossing can be colored in seven or fewer colors, but you can also draw K_7 on the torus so there is a graph which needs at least seven. Surprisingly this was known well before the fact that the plane (which we would expect to be the simplest case) needs at most four colors, and also the proof is much simpler and does not require computers.

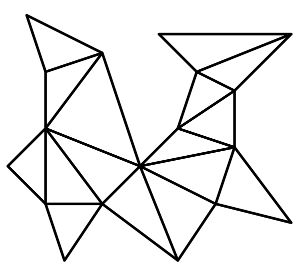
Finally, we give an application for coloring. The art gallery problem asks the following question. Suppose that we have an art gallery which is in the shape of a polygon with n sides (we will assume that there are no holes). What is the minimal number of guards which need to be positioned inside the art gallery so that the entire gallery is under surveillance. If the polygon is convex then we only need one guard who can be positioned anywhere. But the layout for the art galleries might not be arranged in a convex polygon, for example consider the gallery below which has 18 sides.



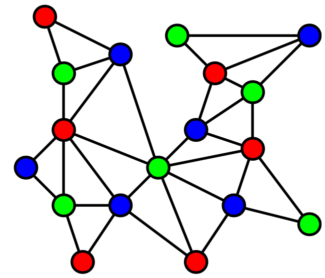
In the art gallery problem for a polygon with n sides with no holes then you need no more than $\lfloor n/3 \rfloor$ guards to protect the gallery.

The notation $\lfloor n/3 \rfloor$ means that you take $n/3$ and round down. So for instance we know that in the gallery above we need at most 6 guards. We also note that there are examples where we need at least $\lfloor n/3 \rfloor$ guards to protect the gallery (we do not give them here) so this result is the best possible.

To prove this we first take out gallery and triangulate it, i.e., add edges between the corners so that the resulting floor layout is a collection of triangles. In general there are many ways to do this (it is not immediately obvious that this can be done, it is not difficult to show, but we will omit that step). One example for the gallery shown above is the following.



We now think of this as a graph where the lines are edges and the vertices are polygons. We claim that we can color the corners in three colors so that in any triangle there is one red corner, one blue corner and one green corner. This is easily seen for the case when we start with three sided polygon. Inductively we can take an interior edge and split the gallery into two halves. By induction we know that we can color each half satisfying the requirement and then we can combine the two halves to get the desired coloring. For example for the gallery above we get (the essentially unique) desired coloring below. (Here essentially unique means that given a triangulation there is only one coloring up to permutation of colors, different triangulations lead to different colorings.)



Now we choose the color that is used least often, since there are n corners to color and we only use three colors there is some corner used at most $\lfloor n/3 \rfloor$ times. Position the guards at the vertices with the color used least often. Note that each triangle has a corner with each color and so each triangle is now under surveillance. But of course once all of the triangles are under surveillance the entire gallery is under surveillance and we are done.

In our case it turns out that based on our triangulation each color was used exactly 6 times, and so we can position guards at either the corners with the blue vertices, the corners with the red vertices, or the corners with the green vertices and protect the entire gallery.

Lecture 25 – June 1

The art gallery problem from the last lecture has a nice application.

Fary's Theorem:
Every planar graph can be drawn in the plane with no edges crossing and as all edges as straight lines.

Previously we said that a planar graph is one which can be drawn in the plane so that no two edges intersect, but we make no requirement about the shape of the edges, so they can bend and go around as needed. This says that every planar graph can be drawn so that all the edges are straight. We sketch the proof of this result here.

First we note that we can limit ourselves to the case that we are dealing with maximally planar graphs, i.e., these are graphs where the addition of any edge would make the graph nonplanar. The important thing about these graphs is that each face is a triangle (if not then we could add edges inside the face to triangulate it and still stay planar). We fix one triangle to act as the outer triangular face and we now proceed by induction to show that we can draw the rest of the graph with straight lines. Clearly, we can draw the triangle with straight lines. Suppose we have shown that we can draw a planar graph with n vertices with straight lines.

Now consider the case with $n+1$ vertices. We know that there is some vertex of degree ≤ 5 in the graph (and in particular we can also assume it is not one of the vertices involved in the outer triangular face). Remove the vertex and the incident edges to give a graph on n vertices, and we now triangulate the face (note that the face will be either a triangle, a quadrilateral or a pentagon). By induction we can now embed the graph on n vertices in the plane with all edges as straight lines. Finally, remove the edges that we added in earlier, we now have to position the last vertex inside the face so that it can connect to all of the other edges using a straight line. This can be done since the face has at most five sides and so by the art gallery problem we know that we need at most $\lfloor 5/3 \rfloor = 1$ guards to keep the entire gallery under surveillance, i.e., there is some point which sees all of the corners, put the $(n+1)$th vertex in that point and connect with a straight edge to all other vertices. We now have our desired straight line embedding for our graph on $n+1$ vertices.

An important type of graph are *trees*. There are a few different ways to characterize a tree. A tree is a graph on n vertices which

- is connected and acyclic (no cycles); or
- is connected and has $n-1$ edges; or
- is acyclic and has $n-1$ edges; or
- between any two vertices there is a unique path.

There are many others as well. We note that for a graph on n vertices to be connected we need at least $n-1$ edges so that in some sense these are the "smallest" graphs on n vertices.

Trees are frequently used to model series of decisions, where we start at a root vertex and then have several options, then for each option we might have several more options and so on until we get to the end of the process. One important example of this are binary trees where we have a root vertex and then every vertex has 0, 1 or 2 vertices coming down from that vertex (only in computer science do trees grow down), which are labeled left or right. The binary comes from the number of outcomes being at most two.

Trees get their names because they do bear a resemblance to actual trees in the sense that trees have nodes branching off into smaller branches which then branch off and (unless through some devious topiary) the tree does not come back in onto itself. An important terminology about trees are leaves. These are the vertices of degree 1 in a tree (so the end of the tree).

Every tree on $n \geq 2$ vertices has at least one leaf.

To prove this we make use of an averaging argument.

Given numbers $n_1, n_2, \ldots, n_k$ let $\overline{n}$ denote their average. Then there is some i and j so that $n_i \leq \overline{n}$ and $n_j \geq \overline{n}$.

In other words, unlike the children in Lake Wobegon, not all the numbers can be above average. In our case let us note that since the graph is connected the degree of each vertex is at least one, i.e., $d(v) \geq 1$ for all v in the tree. By considering the average of the degrees we have that for some vertex

$$d(v) \leq \frac{\displaystyle\sum_{v \in V} d(v)}{n} = \frac{2(n-1)}{n} = 2 - \frac{2}{n} < 2.$$

Here we used the fact that the sum of the degrees is twice the number of edges and that in a tree there are $n-1$ edges. So this shows that some vertex has degree less than 2, which combined with the fact that all vertices have degree at least 1 we can conclude that there is a vertex with degree exactly one, i.e., a leaf.

We can actually say a little more. In particular, if a graph has a vertex of degree m then it has at least m leaves. One way to see this is we start at the vertex of degree m we go out along each edge and we keep walking until we hit a leaf, since there are no cycles each leaf that we end up at is distinct and so we must have at least m leaves.

Alternatively, suppose that we have k leaves in the graph, so that the remaining $n-k-1$ vertices have degree at least 2 (the k leaves and the 1 vertex of degree m). Then we have

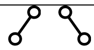
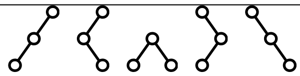
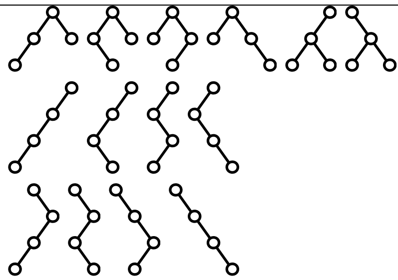
$$2(n-1) = \sum_{v \in V} d(v) \geq k + m + 2(n-k-1),$$

which simplifying gives $k \geq m$, the desired result.

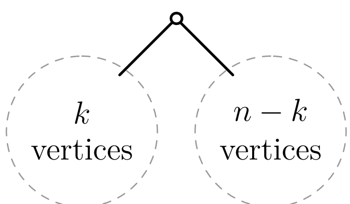
Lecture 26 – June 3

In the last lecture we discussed binary trees. These are trees with a rooted vertex and every vertex has 0, 1 or 2 children where each child is labeled left or right. (Binary here reflects that we have two options at each stage.) A natural question to ask is how many binary trees are there, before we do that we need to say that two binary trees are the same if and only if when we map the root of one tree to the other the trees and labeling (i.e., left/right) remain the same. To get an idea we can look at the first few cases (these are shown below).

In particular we see that we have $1, 1, 2, 5, 14, \ldots$ binary trees. These numbers might look familiar, and they should, they are the Catalan numbers!

n	trees	# of trees
0		1
1		1
2		2
3		5
4		14

To see why we should get the Catalan numbers we note that to make a binary tree on $n+1$ vertices we have a root and then on the left child we have a binary tree on k vertices and on the right child we have a binary tree on $n-k$ vertices (see picture below).



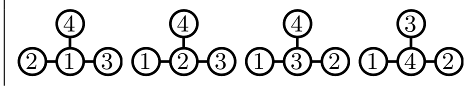
In particular if we let B_n be the number of binary trees on n vertices then we have B_k choices for the binary tree on the left and B_{n-k} choices for the binary tree on the right or $B_k B_{n-k}$ ways to make a binary tree. But then we can also let $k = 0, 1, \ldots, n$ and so we add them all up to get all the possible binary trees on $n+1$ vertices, i.e.,

$$\begin{aligned} B_{n+1} &= B_0 B_n + B_1 B_{n-1} + \cdots + B_n B_0 \\ &= \sum_{k=0}^{n} B_k B_{n-k}. \end{aligned}$$

Note that this is an example of convolution and whenever we get a recurrence of this type it can often be related back to the Catalan numbers.

More generally we can ask how many trees on n vertices there are. We will not answer that question (since it is not trivial!), instead we will look at another problem, how many *labelled* trees on n there are. A labelled tree on n vertices is a tree where we have assigned a label to each vertex, we will use the labels $1, 2, \ldots, n$. Two labeled trees are the same if and only if vertices labeled i and j are adjacent in one tree if and only if they are adjacent in the other tree for all pairs i and j.

Again we can start by listing them all for a few small values. We have the following.

n	trees	#
1	(1)	1
2	(1)—(2)	1
3	(1)—(2)—(3) (1)—(3)—(2) (2)—(1)—(3)	3
4	(1)-(2)-(3)-(4) (1)-(2)-(4)-(3) (1)-(3)-(2)-(4) (1)-(3)-(4)-(2) (1)-(4)-(2)-(3) (1)-(4)-(3)-(2) (2)-(1)-(3)-(4) (2)-(1)-(4)-(3) (2)-(3)-(1)-(4) (2)-(4)-(1)-(3) (3)-(1)-(2)-(4) (3)-(2)-(1)-(4) 	16

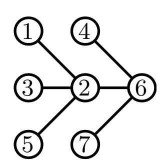
Certainly the first thing we notice is that there are a lot of labelled trees. For instance for $n = 5$ there would be 125 (which explains why we stopped at 4). It turns out that there is a simple expression for the number of labelled trees on n vertices.

> *Cayley's Theorem:*
> There are n^{n-2} labelled trees on n vertices.

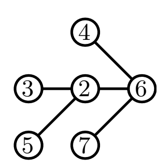
This is certainly a simple expression and the form n^{n-2} is highly suggestive. It is not hard to find something else that has the form n^{n-2}, namely consider words of length $n-2$ where each letter in the word is in $\{1, 2, \ldots, n\}$. One method to prove Cayley's Theorem is to show that there is a one-to-one correspondence between labelled trees on n vertices and these words of length $n-2$. Conveniently enough there is a nice correspondence known as Prüfer Codes.

> *Prüfer Codes:*
> Given a labelled tree on n vertices we construct a word using the following procedure. For the current tree find the *leaf* which has the lowest label, remove the leaf and write down the label of the vertex that it was adjacent to. Repeat this until there are no leafs.

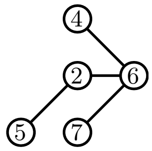
As an example, consider the following labelled tree on 7 vertices.



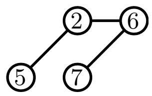
The lowest labelled leaf is 1 and it is adjacent to 2. So we remove the leaf and record 2. We now have the tree.



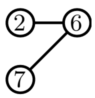
The lowest labelled leaf is now 3 and it is adjacent to 2. So we remove the leaf and record 2, so our word is currently 22. We now have the tree.



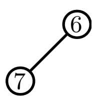
The lowest labelled leaf is now 4 and it is adjacent to 6. So we remove the leaf and record 6, so our word is currently 226. We now have the tree.



The lowest labelled leaf is now 5 and it is adjacent to 2. So we remove the leaf and record 2, so our word is currently 2262. We now have the tree.



The lowest labelled leaf is now 2 and it is adjacent to 6. So we remove the leaf and record 6, so our word is currently 22626. We now have the tree.



The lowest labelled leaf is now 6 and it is adjacent to 7. So we remove the leaf and record 7, so our word is currently 226267. We now have the tree.

This tree has no leaves (note that the only tree without leaves are the trees on one or fewer vertices), and so we stop. So this tree is associated with the word 226267. Now at this point we notice that this has 6 letters. In particular, this procedure produces a word of length $n-1$, and since each letter in the word is a label on a vertex each letter is one of $\{1, 2, \ldots, n\}$.

We wanted to have a word of length $n-2$ not $n-1$, but let us make an observation. In the last lecture we showed that every tree on $n \geq 2$ vertices has a leaf, in fact it is easy to show that every tree on $n \geq 2$ vertices has at least two leaves. Since we will always be taking the *smallest* leaf out we will never take out n as we build our code. In particular, the last letter in the sequence of length $n-1$ must *always* be n, so we can ignore it and just work with the first $n-2$ digits.

We see that for every tree that we can make a Prüfer code, but to be complete we also need to check that there is a one-to-one correspondence. In particular given a sequence of length $n-1$ where the first $n-2$ digits are in $\{1, 2, \ldots, n\}$ and the last digit is n we need to be able to construct the tree that it came from.

> *Deconstructing Prüfer Codes:*
> Given a sequence
> $$a_1 a_2 \cdots a_{n-2} a_{n-1}$$
> where $a_i \in \{1, 2, \ldots, n\}$ for $i \leq n-2$ and $a_{n-1} = n$ construct the sequence $b_1 b_2 \cdots b_{n-1}$ recursively by letting
> $$b_i = \min\big\{k \mid k \notin \{b_1, \ldots, b_{i-1}, a_i, \ldots, a_{n-1}\}\big\}.$$
> Then form a tree on n labeled vertices with edges joining a_i and b_i for $i = 1, 2, \ldots, n-1$.

The idea is that the Prüfer code was recording the vertices adjacent to the leaf that we removed, if we knew the corresponding labels of what was removed then we could form all the edges of the tree. So what we are doing with the b_i is determining the labels of the leafs that were removed. The idea being that b_i cannot be one of $b_1, \ldots, b_{i-1}$ since those vertices were already removed and it cannot be one of $a_i, \ldots, a_{n-1}$ since they need to stay in the graph after the current step, so then b_i must be the smallest number not on the list.

Let us work back through our example above. Applying the above procedure to the word 226267 we see that the smallest number not seen is 1 so we have $b_1 = 1$.

a_i :	2	2	6	2	6	7
b_i :	1					

We now cover up a_1 and see that the smallest number not seen is 3 so $b_2 = 3$.

a_i :	~~2~~	2	6	2	6	7
b_i :	1	3				

We now cover up a_2 and see that the smallest number not seen is 4 so $b_3 = 4$.

a_i :	~~2~~	~~2~~	6	2	6	7
b_i :	1	3	4			

We now cover up a_3 and see that the smallest number not seen is 5 so $b_4 = 5$.

a_i :	~~2~~	~~2~~	~~6~~	2	6	7
b_i :	1	3	4	5		

We now cover up a_4 and see that the smallest number not seen is 2 so $b_5 = 2$.

a_i :	$\not 2$	$\not 2$	$\not 6$	$\not 2$	6	7
b_i :	1	3	4	5	2	

We now cover up a_5 and see that the smallest number not seen is 6 so $b_6 = 6$.

a_i :	$\not 2$	$\not 2$	$\not 6$	$\not 2$	$\not 6$	7
b_i :	1	3	4	5	2	6

So this says our tree should have edges 2–1, 2–3, 6–4, 2–5, 6–2 and 7–6. Comparing this to our original tree we see that it matches.

We have not checked (nor will we) all of the details but the above two procedures give a one-to-one correspondence between labelled trees on n vertices and sequences of length $n-2$ with letters from $\{1, 2, \ldots, n\}$. So we can conclude that there are n^{n-2} such trees, giving Cayley's Theorem.

Given a graph G a spanning tree of the graph is a subgraph of G which is a tree and which contains all of the vertices of G (i.e., it spans). We make the following observation

> A graph has a spanning tree if and only if it is connected.

To see this we note that if the graph has a spanning tree then since a tree is connected we can get from any vertex to any other vertex using the tree. So we can get from any vertex to any other vertex in the whole graph (since the whole graph contains the tree) so the graph must be connected. On the other hand if the graph is connected we look for any edge that is contained in a cycle, if we find an edge we remove it. We continue until we cannot find such an edge, the resulting graph is a subgraph which is connected and does not have cycles, i.e., a spanning tree.

A *weighted* graph is a graph where each edge has been given a weight. For instance we might have several different routes to take and we can map all the various roads as a graph and on each edge we put the amount of time it would take to travel on that stretch of road. In this case one thing that we might be looking for are the shortest distance connecting two points (i.e., the start and end).

As another example we might be trying to hook up cities on a fiber optic network. We could draw a graph where the edges are the possible locations for us to lay fiber between cities and each edge is the cost associated with connecting the cities. A reasonable problem to ask is what is the cheapest way to connect the cities. Well we might as well assume that we remove redundancies since we can route data through the fibers we do not need to have any cycles so we are looking for a tree. In particular we want to have the tree that would give us the lowest total cost. This is an example of the minimum spanning tree problem.

Given a weighted graph G the minimum spanning tree problem asks us to find the spanning tree with the lowest total sum of edge weights. There are several different ways to do this, but they all boil down to one basic idea: *be greedy.*

A greedy algorithm is one where at each stage we make the best possible choice. For instance when you get change back at a store the cashier starts with the largest bill less than what you are owed and gives you that and then keeps repeating the procedure until the account is settled. The advantage of a greedy algorithm is that it is simple to understand and easy to implement. The disadvantage is that is is not always the best option, but in this case we are in luck! Below we will give two different algorithms that solves the minimum spanning tree problem.

> *Finding a minimum spanning tree (I):*
> Given a connected weighted graph look for the "heaviest" edge whose removal does not disconnect the graph. Continue the process until we have a tree.

Note that this is similar to the same proof showing that every connected graph has a spanning tree, the only difference is that we now make sure we remove the heaviest edge. So we start with all the edges and we trim down. Another algorithm works in the opposite direction by building up edges.

> *Finding a minimum spanning tree (II):*
> We start with no edges and we look among all the edges in G not currently in and add the "lightest" edge whose addition does not create a cycle. Continue the process until we have a tree.