More practice problems from 31A

14 variations on a theme

> Slopes measure rate of change (secant lines give average rate of change, tangent lines give instantaneous rate of change).

Example: Find the average rate of change to the curve $y = x^2$ from $x = a$ to $x = b$.

Example: Find the instantaneous rate of change to the curve $y = x^2$ at $x = (a + b)/2$.

(See also PM1.6, M1.5)

> Finding tangent lines is important since we can use it to get information about the function.

Example: Find the tangent line to $y = \sin(\pi x^2) - \pi x$ at $x = 1$.

Example: For $f(x) = 2x^3 - 12x^2 + 7x + 2$ the y-intercept is 2. Find the (unique) $x \neq 0$ so that the y-intercept of the tangent line is also 2.

(See also PM1.3, PM1.5, M1.1, M1.4, PFB.4, PFC.4)

> If two quantities are related then their rates of change are also related.

Example: A train is heading east towards Junction City at a speed of 15 miles and is currently 30 miles away, at the same time a second train is heading south away from Junction City at a speed of 10 miles and is currently 15 miles away. At what rate are the two trains moving apart from each other?

Example: If applying a force to a wrench at a 90° angle the amount of torque τ produced is $\tau = rF$ where r is the length along the wrench you apply the force and F is the amount of force applied. Suppose that you need to produce a constant torque of 50 Newton meters. Find dr/dt if the amount of force you can generate is changing at a rate of $dF/dt = -5$ Newtons per second when $F = 100$ Newtons.

(See also PM2B.3, M2.3, PFA.2, PFB.2, PFC.3)

> Even if we don't have an explicit relationship, we can still find the derivative.

Example: Find $\dfrac{dy}{dx}$ if $xy^3 + 7y = 3x + 5x^2y^2$. Show that $(1, 1)$ is a critical point on this curve.

Example: An implicit function has a *vertical* tangent line if $dx/dy = 0$. Find the *four* points on the implicitly defined function $2y^3 + 3y^2 = 4x^2 + 5x + 1$ where we have a vertical tangent line.

(See also PM2A.1, PFB.2, PFC.4)

> If we cannot get an exact value, derivatives/tangent lines can help us to get a good estimate.

Example: Approximate $(10.07)^5$.

Example: For $y = 3x \sin x$, if $x = (\pi/4) \pm 0.05$, estimate the range of corresponding y values.

(See also PM2A.2, PM2B.1, M2.2, PFA.3, PFB.4)

> We can use derivatives to look for and/or identify maximum/minimum values.

Example: Find the global minimum and maximum of $h(s) = 2s^3 + 3s^2 - 12s - 18$ for $0 \leq s \leq 3$.

Example: Find the *two* inflection points for $y = \theta^2 + \sin(2\theta)$ for $0 \leq \theta \leq \pi$, also identify the intervals where the function is concave up and where the function is concave down.

(See also PM2A.3, PM2A.1, PM2A.4, PM2B.2, PM2B.5, M2.4, M2.5, PFA.4, PFA.8, PFB.3, PFC.2)

> The Fundamental Theorem of Calculus tells us that integration and differentiation are two sides of the same coin.

Example: Given $F(x) = \displaystyle\int_{2x^2}^{x^3} \sqrt{2 + \sqrt[3]{t}}\, dt$, find the tangent line to $F(x)$ at $x = 2$.

Example: Suppose that $G(x)$ is the anti-derivative of $g(x)$ and that $G(-x) = G(x)$. Show $g(-x) = -g(x)$. (Hint: evaluate $\displaystyle\int_{-x}^{x} g(t)\, dt$ and then take the derivative of both sides.)

(See also PM2A.5, PFA.5, PFA.8, PFB.6, PFB.7, PFC.6)

> We can rewrite integrals by combining them and/or changing the order of integration. Similarly, we can break integrals into pieces and work on each piece separately.

Example: Find $\displaystyle\int_1^2 f(x)\, dx$, given that

$$\int_0^1 2f(2x)\, dx + \int_1^3 f(x)\, dx = \int_0^3 f(x)\, dx.$$

Example: For $x \geq 0$, find $F(x) = \displaystyle\int_0^x 6|t^2 - t|\, dt$. (Hint: find $F(x)$ as a piecewise function according to how we can break up the function inside the integral.)

(See also PFA.1, PFA.6, PFB.9)

Rewriting integrals turns hard problems into easier problems. (Particularly true when dealing with polynomials and/or trigonometric functions.)

Example: Find $\displaystyle\int \frac{1}{\big(\sin\theta+\cos\theta\big)^2}\,d\theta$.

Example: Find $\displaystyle 5\int_1^4 \frac{(x+1)(x-1)}{\sqrt{x}}\,dx$.

(See also PFB.5, PFC.5)

Substitution is one of the best tools for simplifying integrals. Look for functions inside of functions.

Example: Find $\displaystyle\int_0^1 \sqrt{\frac{1}{x}-1}\,dx$. (Hint: try $u=\sqrt{x}$.) (Don't worry about the "∞" at 0, just go for it!)

Example: Find $\displaystyle\int \frac{\sin(\sqrt{x})\sin(2\sqrt{x})}{\sqrt{x}}\,dx$.

(See also PFA.6, PFA.7, PFB.5, PFC.5, PFC.9)

The Mean Value Theorem tells us that at some time we are average.

Example: Find the average value of $g(x)=x^2+2x+6$ on the interval $0\le x\le 2$ and find $0\le c\le 2$ such that $g(c)$ is the average value.

Example: Suppose that $H(x)$ is the anti-derivative of $h(x)$ (with $h(x)$ continuous) and for all x, $H(x+p)=H(x)$ (i.e., H is periodic with period p). Show that for *any* a there is some c with $a\le c\le a+p$ with $h(c)=0$.

(See also PM2B.4, M2.1, PFB.8, PFC.8)

If we know how fast we were changing, then we know how much we have changed.

Example: Looking over his attendance records Professor Butler has discovered as the term progressed that fewer students came to lecture (thus breaking his super-sized heart). In particular, he noticed that t weeks into the quarter that there were $200-20t+t^2$ students per lecture. Find the average number of students per lecture during the ten week quarter (i.e., from $t=0$ to $t=10$).

Example: Given $\displaystyle f'(x)=\frac{1}{x^2+1}+\sec^2 x$ and $f(0)=3$, find $f(x)$.

(See also PFC.7)

Integration can find area, and by finding area we can do some integrals.

Example: Find $\displaystyle\int_0^2 \sqrt{t^4+9}\,dt+\int_3^5 \sqrt[4]{t^2-9}\,dt$. (Hint: interpret these two integrals as area and show how the areas combine to form a simple shape.)

Example: Find the area bounded by $f(x)=2\sin x$ and $g(x)=\sec x\tan x$ between $x=0$ and the smallest $x>0$ where the two curves intersect.

(See also PFA.8, PFA.9, PFB.1, PFB.7, PFC.1)

Integration can find volume, sometimes we can compute the same volume in two different ways.

Example: Let $f(x)\ge 0$ for $1\le x\le 4$ find

$$\int_1^4 \big(f(x)-x\big)^2\,dx,$$

given that when we rotate the region between $f(x)$ and the x-axis between $x=1$ and $x=4$ around the x-axis the resulting volume is 32π while if we rotate the same region around the y-axis the volume is 48π.

Example: Show that

$$\pi\int_0^1 \sqrt{1-t^4}\,dt=2\pi\int_0^1 t\sqrt[4]{1-t^4}\,dt.$$

(Hint: show how both integrals give the volume of the *same* object.)

(See also PFA.10, PFB.10, PFC.10)