MATH 31A (Butler)
Midterm II, 19 February 2010

1. Find the unique value c that satisfies the Mean Value Theorem for the function $f(x) = \arctan(\sin x)$ for x between $a = 0$ and $b = \pi/2$. (Your answer will involve an arcsin or arccos term. Hint: $\sin^2 x + \cos^2 x = 1$; the quadratic formula is awesome!)

First we recall the statement of the Mean Value Theorem, namely if our function is continuous and differentiable between points a and b then there is some c between a and b so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In our case we need to have

$$\begin{aligned} f'(c) &= \frac{f(\frac{\pi}{2}) - f(0)}{\frac{\pi}{2} - 0} \\ &= \frac{\arctan(\sin\frac{\pi}{2}) - \arctan(\sin 0)}{\frac{\pi}{2}} \\ &= \frac{\arctan(1) - \arctan(0)}{\frac{\pi}{2}} = \frac{\frac{\pi}{4} - 0}{\frac{\pi}{2}} = \frac{1}{2}. \end{aligned}$$

Taking the derivative of the function (using the chain rule and the rules for the derivatives of the sine and arctangent functions) we have

$$f'(x) = \frac{1}{1 + (\sin x)^2} \cdot \cos x = \frac{\cos x}{1 + \sin^2 x}.$$

So we need

$$f'(c) = \frac{\cos c}{1 + \sin^2 c} = \frac{1}{2} \quad \text{or} \quad 2\cos c = 1 + \sin^2 c.$$

Using $\sin^2 c + \cos^2 c = 1$ this can be rewritten as

$$2\cos c = 2 - \cos^2 c \quad \text{or} \quad \cos^2 c + 2\cos c - 2 = 0.$$

This is a quadratic (but with $\cos c$ instead of x) so we can use the quadratic formula to find

$$\cos c = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-2)}}{2} = \frac{-2 \pm \sqrt{12}}{2} = -1 \pm \sqrt{3}.$$

But $-1 - \sqrt{3} < -1$ and so can never be the cosine of an angle. So we must have that $\cos c = -1 + \sqrt{3}$ or

$$c = \arccos(-1 + \sqrt{3}).$$

2. You have recently been hired as the chief architect for one of the pyramids constructed by Pharaoh Sneferu in ancient Egypt. After some consultation the pharaoh has agreed to a pyramid design that is 500 cubits wide and 300 cubits high (a cubit is the system of measurement used in ancient Egypt). The volume of a pyramid is $\frac{1}{3}b^2h$ where b is the length of one side of the base and h is the height; so that the pyramid will require 25,000,000 cubic cubits of stone. After getting in touch with your stone contractor you discover that there are only 23,000,000 cubic cubits of stone available. The pharaoh gives the go ahead to build a (slightly) smaller pyramid, but with the same proportions as before.

Estimate how many cubits smaller the base of the pyramid will end up being.

We have that the volume of the pyramid is

$$V = \frac{1}{3}b^2h,$$

but we can rewrite it as the single variable b since we know that we want to keep the same proportions. In particular we want to still have

$$\frac{h}{b} = \frac{300}{500} = \frac{3}{5} \quad \text{so} \quad h = \frac{3}{5}b.$$

Putting this in we have that the volume is

$$V(b) = \frac{1}{3}b^2 \cdot \frac{3}{5}b = \frac{1}{5}b^3.$$

We want to estimate the change in the base (Δb) given that we have a change in the volume (ΔV). We know that they are related by

$$\Delta V \approx V'(b)\Delta b.$$

Since $V'(b) = \frac{3}{5}b^2$ we can conclude that

$$\underbrace{-2,000,000}_{=\Delta V} \approx V'(500)\Delta b.$$

Computing we have

$$V'(500) = \frac{3}{5}(500)^2 = 150,000.$$

So we can conclude that

$$\Delta b \approx \frac{\Delta V}{V'(500)} = \frac{-2,000,000}{150,000} = -\frac{200}{15} = -\frac{40}{3} \text{ cubits.}$$

3. While studying for this exam some students decide to take a break and bake brownies. They stir the brownie batter in a bowl which has a hemispherical shape with a radius of 5 inches, they then pour it into the only pan they have which is a large circular cake pan (i.e., the shape of a cylinder) which is 12 inches in diameter and 2 inches deep.

"I couldn't help but notice that when you were pouring the brownie batter into the pan that you were able to pour it in so that the depth of the batter in the pan raised at a constant rate of 1/6 of an inch per second," one of the students observed.

"As the height in the pan went up, then the height in the bowl must have been going down. Did you see how fast the depth of the batter in the mixing bowl was dropping?" the other student replied.

"I wasn't paying attention to the mixing bowl, but we can figure it out."

"Oh come on. I just want brownies, there is *no way* Professor Butler would ever ask that on the midterm!"

Find the rate at which the depth of the batter is falling in the mixing bowl when the depth of the batter in the mixing bowl is 3 inches (include units). (Hint: the volume of batter of depth h in a hemispherical bowl with radius r is $\pi(rh^2 - \frac{1}{3}h^3)$.)

That crazy Professor Butler!
If we think about what is happening we have that batter is being transferred from one container to the next; but there is always the same amount (i.e., same volume) of batter. So that the volume of the batter in the mixing bowl plus the volume of the batter in the cake pan is a constant, let us call it V. The hint gives us the volume of the batter in the bowl in terms of the depth h of batter in the bowl and in our case $r = 5$. The volume of the batter in the cake pan is the volume of a cylinder, i.e., $\pi R^2 q$ where R is the radius of the cake pan (in our case 6) and q is the depth of the batter in the cake pan. So we are told $\frac{dq}{dt} = \frac{1}{6}$ and we want to find $\frac{dh}{dt}$ when $h = 3$, so this seems to be a clear case of a related rates problem. So first we set up our relationship

$$\underbrace{\pi(5h^2 - \frac{1}{3}h^3)}_{\text{bowl}} + \underbrace{36\pi q}_{\text{pan}} = V$$

Now we take the derivative of both sides (remember that V is a constant) to find

$$10\pi h\frac{dh}{dt} - \pi h^2\frac{dh}{dt} + 36\pi\frac{dq}{dt} = 0 \quad \text{or} \quad (10h - h^2)\frac{dh}{dt} = -36\frac{dq}{dt}.$$

Plugging in our values for h and $\frac{dq}{dt}$ we have

$$21\frac{dh}{dt} = -36\cdot\frac{1}{6} = -6 \quad \text{so} \quad \frac{dh}{dt} = -\frac{6}{21} = -\frac{2}{7}\,\frac{\text{inches}}{\text{second}}.$$

[Note: we did not need to know how much batter there was to answer the question, or use the depth of the cake pan.]

4. Your Super Bowl party was a success and now you find yourself planning for the biggest sporting event of the year; the women's gold medal match in curling (which takes place on February 26 at 3:00pm).

Past analysis of previous parties has led to the development of a chip index where the higher the chip index the better the party. In particular, the chip index is N^2P where N is the number of nacho chip bags that you have and P is the number of potato chip bags that you have.

Given that you have \$18 for your chip fund and a bag of nacho chips cost \$3 and a bag of potato chips cost \$1, how many bags of each chip should you buy to maximize the chip index and thus have the best party.

We want to maximize N^2P, but this has two variables N and P both of which are greater than or equal to zero (we cannot have negative bags of chips!). However, we know that N and P are related, namely that the total cost of buying our combination of chips is \$18. So we have that

$$3N + P = 18 \quad \text{or} \quad P = 18 - 3N.$$

Therefore our chip index as a function of a single variable is

$$c(N) = N^2(18 - 3N) = 18N^2 - 3N^3.$$

We now want to maximize this function and so we take the derivative and look for critical points. We have

$$c'(N) = 36N - 9N^2 = 9N(4 - N)$$

so that we have critical points at $N = 0$ and $N = 4$. Taking the second derivative we have

$$c''(N) = 36 - 18N,$$

so that $C''(0) = 36$ showing that $N = 0$ corresponds to a minimum and $C''(4) = -36$ showing that $N = 4$ corresponds to a maximum, and is our desired solution. We then also have that $P = 18 - 3{\cdot}4 = 6$.

So we should buy four bags of nacho chips and six bags of potato chips to maximize our chip index.

5. Let $g(x)$ be the antiderivative of $f(x) = x + 5 - 2\sqrt{x^2+3}$ such that $g(0) = -4$. (Hint: when x is large (positive or negative) then $\sqrt{x^2+3} \approx |x|$.)

(a) Find the x values for the critical point(s) of $g(x)$. Also, use either the first derivative or second derivative test to determine whether the critical point(s) is a maximum, a minimum or neither. (Clearly indicate which test you are using.)

To find the critical points we need to look at $g'(x)$, but since $g(x)$ is the antiderivative of $f(x)$ we have that $g'(x) = f(x) = x+5-2\sqrt{x^2+3}$. This function is always defined so any critical point occurs where

$$2\sqrt{x^2+3} = x+5 \quad \text{or} \quad 4(x^2+3) = x^2+10x+25 \quad \text{or} \quad 3x^2-10x-13=0.$$

This factors to $(3x-13)(x+1) = 0$ so that there are two critical points, $x = -1$ and $x = \frac{13}{3}$ (you could also have used the quadratic formula).

To determine if a critical point is a minimum or a maximum we will use the first derivative test. So we need to plug a value less than -1, between -1 and $\frac{13}{3}$ and larger than $\frac{13}{3}$ into the derivative and see what the signs are. Using the hints for large x, we have

$$\begin{aligned} g'(-100) \approx -100+5-200 < 0 &\quad \longleftarrow \quad \text{so } g(x) \text{ is decreasing for } x \le -1 \\ g'(1) = 2 > 0 &\quad \longleftarrow \quad \text{so } g(x) \text{ is increasing for } -1 \le x \le \tfrac{13}{3} \\ g'(100) \approx 100+5-200 < 0 &\quad \longleftarrow \quad \text{so } g(x) \text{ is decreasing for } x \ge 1 \end{aligned}$$

In particular, since the function is decreasing to the left of -1 and increasing to the right of -1 we have that $x = -1$ is at a minimum; and since the function is increasing to the left of $\frac{13}{3}$ and decreasing to the right of $\frac{13}{3}$ we have that $x = \frac{13}{3}$ is at a maximum.

(b) Find the x values for the inflection point(s) of $g(x)$. Also, indicate on which intervals the function $g(x)$ is concave up and which intervals the function $g(x)$ is concave down.

Continuing from part (a) we first note that using the chain rule

$$g''(x) = f'(x) = 1 - 2\frac{1}{2}(x^2+3)^{-1/2}(2x) = 1 - \frac{2x}{\sqrt{x^2+3}} = \frac{\sqrt{x^2+3}-2x}{\sqrt{x^2+3}}.$$

This is never undefined (since the denominator is always positive. Looking at the numerator this will be 0 if

$$\sqrt{x^2+3} = 2x \quad \text{or} \quad x^2+3 = 4x^2 \quad \text{or} \quad x^2 = 1 \quad \text{or} \quad x = \pm 1.$$

But the solution $x = -1$ doesn't actually work (it snuck in when we squared both sides). So there is only one place where $g''(x) = 0$ and that is at $x = 1$. Plugging points in we have that $g''(-1) = 2 > 0$ showing that $g(x)$ is concave up for $-\infty < x < 1$ and $g''(100) \approx -1 < 0$ showing that $g(x)$ is concave down for $1 < x < \infty$.

Note that in answering these two questions we never had to find $g(x)$ or use that $g(0) = -4$.