Student name: ____________________

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MATH 31A (Butler)

Midterm I, 22 January 2010

This test is closed book and closed notes. No calculator is allowed for this test. For full credit show all of your work (legibly!). Problem 3 is worth 15 pts, the remaining problems are worth 10 points (a total of 55 points).

1. Given $y = 5 - 2x$ is tangent to $f(x)$ at $x = 3$, find the line tangent to the function $g(x) = 3x^2 f(x) - 6$ at $x = 3$.

First we note that to find the tangent line to $g(x)$ at $x = 3$ we need to compute $g(3)$ and $g'(3)$. Using the rules of derivatives we have

$$g'(x) = 6xf(x) + 3x^2 f'(x).$$

So we have the following.

$$\begin{aligned} g(3) &= 3{\cdot}3^2 f(3) - 6 = 27f(3) - 6 \\ g'(3) &= 6{\cdot}3f(3) + 3{\cdot}3^2 f'(3) = 18f(3) + 27f'(3) \end{aligned}$$

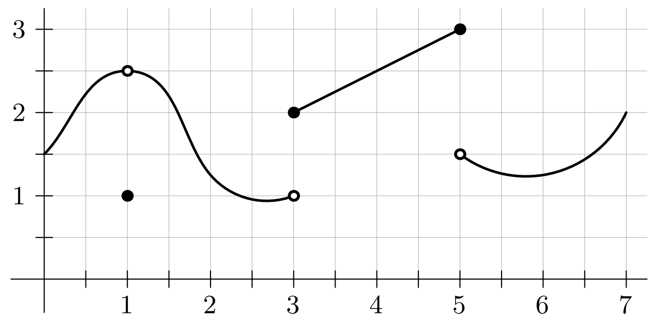
So what we really need are $f(3)$ and $f'(3)$. Fortunately for us we have the tangent line at $x = 3$, in particular we have that $f(3) = 5 - 2{\cdot}3 = -1$ (i.e., the same point as the tangent line) and $f'(3) = -2$ (the slope of the tangent line). So we have the following.

$$\begin{aligned} g(3) &= 27(-1) - 6 = -33 \\ g'(3) &= 18(-1) + 27(-2) = -72 \end{aligned}$$

Finally, the tangent line will be

$$y = g(3) + g'(3)(x - 3) = -33 + (-72)(x - 3) = -72x + 183.$$

2. Let $\kappa(x)$ be the function shown below.



Answer the following questions. (The answer might be that it does not exist.)

(a) $\lim_{x\to 1} \kappa(x) =$

> The limit is 5/2. Since the function from both sides of 1 approach the values of 5/2.

(b) $\kappa(1) =$

> The value of the function is 1. This is seen by the filled in circle at $(1,1)$ on the above figure.

(c) $\kappa'(4) =$

> We have $\kappa'(4) = 1/2$. This is because the function around $x = 4$ looks like a line with slope $1/2$ and so the derivative (which is the slope of the tangent line) will also be $1/2$.

(d) $\lim_{x\to 3} \kappa(x) =$

> This limit does not exist. This is because from the left the function approaches 1 and from the right the function approaches 2. Since the two sided limits don't match the limit does not exist.

(e) $\lim_{x\to 3} \kappa(x)\kappa(x+2) =$

> The limit is 3. To see this again let us look at the one sided limits we have
>
> $$\lim_{x\to 3-} \kappa(x)\kappa(x+2) = \left(\lim_{x\to 3-} \kappa(x)\right)\left(\lim_{x\to 3-} \kappa(x+2)\right) = 1{\cdot}3 = 3$$
> $$\lim_{x\to 3+} \kappa(x)\kappa(x+2) = \left(\lim_{x\to 3+} \kappa(x)\right)\left(\lim_{x\to 3+} \kappa(x+2)\right) = 2{\cdot}\frac{3}{2} = 3$$
>
> Since they both agree and equal 3 then the limit exists and is 3.

3. Evaluate the following limits. (Hint: $\sin^2 x + \cos^2 x = 1$, $\cos(2x) = \cos^2 x - \sin^2 x$, $\sin(2x) = 2\sin x \cos x$, and $\sin A \sin B = \frac{1}{2}\big(\cos(A-B) - \cos(A+B)\big)$.)

(a) Find $\lim\limits_{x\to 2}\left(\dfrac{\sqrt{x+2}-x}{x-2}\right)$.

It is easy to see that this goes to $0/0$ and so we need to manipulate. We start by multiplying the conjugate, and then factor and cancel to get the following.

$$\begin{aligned}\frac{\sqrt{x+2}-x}{x-2} &= \frac{\sqrt{x+2}-x}{x-2}\cdot\frac{\sqrt{x+2}+x}{\sqrt{x+2}+x} = \frac{(x+2)-x^2}{(x-2)(\sqrt{x+2}+x)}\\ &= \frac{-(x^2-x-2)}{(x-2)(\sqrt{x+2}+x)} = \frac{-(x-2)(x+1)}{(x-2)(\sqrt{x+2}+x)} = \frac{-(x+1)}{(\sqrt{x+2}+x)}\end{aligned}$$

And so we have

$$\lim_{x\to 2}\left(\frac{\sqrt{x+2}-x}{x-2}\right) = \lim_{x\to 2}\frac{-(x+1)}{(\sqrt{x+2}+x)} = \frac{-3}{\sqrt{4}+2} = -\frac{3}{4}.$$

(b) Find $\lim\limits_{x\to\pi/4}\left(\dfrac{\cos(2x)}{\cos x - \sin x}\right)$.

It is easy to see that this goes to $0/0$ and so we need to manipulate. Substituting what we know for $\cos(2x)$ and "factoring" we have

$$\begin{aligned}\lim_{x\to\pi/4}\left(\frac{\cos(2x)}{\cos x-\sin x}\right) &= \lim_{x\to\pi/4}\left(\frac{\cos^2 x-\sin^2 x}{\cos x-\sin x}\right)\\ &= \lim_{x\to\pi/4}\left(\frac{(\cos x-\sin x)(\cos x+\sin x)}{\cos x-\sin x}\right)\\ &= \lim_{x\to\pi/4}\big(\cos x+\sin x\big) = \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2} = \sqrt{2}.\end{aligned}$$

(c) Find $\lim\limits_{t\to 0+}\big(\sin t \sin \frac{1}{t}\big)$.

In this limit it goes to 0 times something undefined (since $\sin(1/t)$ does not have a limit). But we recall that the sine function is bounded between -1 and 1. So we have (for $t > 0$)

$$-\sin t \le \sin t \sin\frac{1}{t} \le \sin t.$$

By the Squeeze Theorem since both $\sin t$ and $-\sin t$ go to 0 as t goes to 0 we can conclude that

$$\lim_{t\to 0+}\big(\sin t\sin\frac{1}{t}\big) = 0.$$

4. Let $h(x) = \begin{cases} 2x^3 - 4x + 5 & \text{if } x \le 1; \\ a\sqrt{x} + b & \text{if } x > 1. \end{cases}$

For which values of a and b does the function $h(x)$ have a derivative at $x = 1$?
(Hint: we must have $\lim_{x \to 1-} h(x) = \lim_{x \to 1+} h(x)$ and $\lim_{x \to 1-} h'(x) = \lim_{x \to 1+} h'(x)$.)

What an awesome hint! First we note that away from $x = 1$ we can take the derivatives easily since it is the derivative of the respective parts (remember that for derivatives we only care about what is happening nearby). So we first note that

$$h'(x) = \begin{cases} 6x^2 - 4 & \text{if } x < 1; \\ \frac{1}{2}ax^{-1/2} & \text{if } x > 1. \end{cases}$$

(Remember that when working with $\sqrt{x}$ we should use $x^{1/2}$ instead.)
So now applying the hint we need

$$\begin{aligned} \lim_{x \to 1-} h(x) &= \lim_{x \to 1+} h(x); \quad \text{or} \\ \lim_{x \to 1-} \big(2x^3 - 4x + 5\big) &= \lim_{x \to 1+} \big(a\sqrt{x} + b\big); \quad \text{or} \\ 3 &= a + b. \end{aligned}$$

And we also need

$$\begin{aligned} \lim_{x \to 1-} h'(x) &= \lim_{x \to 1+} h'(x); \quad \text{or} \\ \lim_{x \to 1-} \big(6x^2 - 4\big) &= \lim_{x \to 1+} \big(\frac{1}{2}ax^{-1/2}\big); \quad \text{or} \\ 2 &= \frac{1}{2}a. \end{aligned}$$

From this we see that $a = 4$ and then $b = -1$.

5. The Straw Hat Pirates are setting forth on their ship the *Going Merry* between two islands, the trip takes a total of five days. Their navigator, Nami, notes that the distance away from the starting island after t days is $q(t) = 9t^2 - t^3 + 3t$ miles.

(a) What is the average rate of speed of their ship from the day they set out ($t = 0$) to the third day of the trip ($t = 3$)? (Include units!)

The average rate of speed over the first three days is

$$\frac{q(3) - q(0)}{3 - 0} = \frac{63 - 0}{3} = 21 \ \frac{\text{miles}}{\text{day}}.$$

(b) How fast is the ship going (i.e., the instantaneous speed) on the second day of the trip ($t = 2$)? (Include units!)

The instantaneous speed (i.e., velocity) is found by the derivative. We have that

$$q'(t) = 18t - 3t^2 + 3,$$

and so the speed that the ship is going on the second day is

$$q'(2) = 18 \cdot 2 - 3{\cdot}2^2 + 3 = 27 \ \frac{\text{miles}}{\text{day}}.$$

(c) On the second day ($t = 2$) is the ship speeding up or slowing down? Justify your answer.

We can tell whether we are speeding up or slowing down by looking at the acceleration, which is the second derivative of $q(t)$. In particular, the acceleration is

$$q''(t) = 18 - 6t.$$

Since $q''(2) = 6$ that tells us the acceleration is positive so that on the second day the ship is speeding up.