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MATH 31A (Butler)

Final, 18 March 2009

This test is closed book and closed notes. No calculator is allowed for this test. For full credit show all of your work (legibly!). Each problem is worth 10 points.

1. Find the area between the curves $y = |1 - x^2|$ and $y = 7 - x^2$.

Since we are dealing with an absolute value, it is better to break it up so that we can drop the absolute value. In particular, we note

$$|1-x^2| = \begin{cases} x^2-1 & \text{if } x \le -1; \\ 1-x^2 & \text{if } -1 \le x \le 1; \\ x^2-1 & \text{if } 1 \le x. \end{cases}$$

So when we integrate we will similarly break our integral into pieces and integrate each corresponding piece. It is also easy to see that the two points of intersection for this curve occur when $x^2 - 1 = 7 - x^2$ or $x^2 = 4$ or $x = \pm 2$. So to find the area we will integrate from -2 to 2 and since in that range we have $7 - x^2 \ge |1 - x^2|$ then we know $7 - x^2$ will be on top. So we have

$$\text{Area} = \int_{-2}^{2} \big((7-x^2) - |1-x^2|\big)\,dx =$$

$$\int_{-2}^{-1} \big((7-x^2)-(x^2-1)\big)\,dx + \int_{-1}^{1} \big((7-x^2)-(1-x^2)\big)\,dx + \int_{1}^{2} \big((7-x^2)-(x^2-1)\big)\,dx$$

$$= \int_{-2}^{-1} \big(8-2x^2\big)\,dx + \int_{-1}^{1} 6\,dx + \int_{1}^{2} \big(8-2x^2\big)\,dx$$

$$= \big(8x - \frac{2}{3}x^3\big)\Big|_{x=-2}^{x=-1} + \big(6x\big)\Big|_{x=-1}^{x=1} + \big(8x - \frac{2}{3}x^3\big)\Big|_{x=1}^{x=2}$$

$$= \big((-8+\frac{2}{3}) - (-16+\frac{16}{3})\big) + \big((6)-(-6)\big) + \big((16-\frac{16}{3}) - (8-\frac{2}{3})\big) = \frac{56}{3}.$$

(Note: one way to simplify the computations is to note that both of these functions are symmetric around the y-axis (i.e., both even). So we could also have done the integral from 0 to 2 and doubled the result.)

2. From a small island in the little of a large lake you set out in a canoe and head due east and at the same time your friend sets out in a canoe and heads due north. After a few hours your friend calls you up on a walkie-talkie to ask how you are doing, and also to find out how far you have gone.

"I am doing fine, but I have not kept track so I don't know how far I have gone. However, I am making good time and right now I am going at a speed of one mile per hour" you tell your friend.

"That's good, I've gone a few miles and right now I'm only doing one third of a mile per hour."

You glance down and notice that your walkie-talkie has built in sensors which indicate that your friend's walkie-talkie (and so also your friend) is currently five miles away from you and you are moving apart from each other at a speed of one mile per hour. After a few mental calculations you call your friend back.

"I now know how far I have gone, and I can also tell you how far you have gone."

Find the distance that you have travelled in your canoe as well as the distance your friend has traveled in the canoe.

Since we have a lot of rates given to us this seems to have something to do with related rates. But curiously we aren't asked to find a rate, but the relationship between the rates can be used to help us identify the correct values.
So let x be the distance that you have canoed and let y be the distance that your friend has canoed and let z be the distance between the two of you. Since you sailed due east and your friend sailed due north then x, y and z form the side lengths of a right triangle. In particular we have

$$x^2 + y^2 = z^2.$$

We also know that $\frac{dx}{dt} = 1$ (your speed), $\frac{dy}{dt} = \frac{1}{3}$ (your friend's speed), $z = 5$ (distance between you) and $\frac{dz}{dt} = 1$ (rate at which you are moving apart). Taking the derivative of both sides with respect to t we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}.$$

First we can divide everything by 2, and now we can plug in everything that we know to get

$$x + \frac{1}{3}y = 5 \quad \text{or} \quad x = 5 - \frac{1}{3}y.$$

Substituting this back into the original equation, along with $z = 5$ we have

$$\left(5 - \frac{1}{3}y\right)^2 + y^2 = 5^2 \quad \text{or} \quad 25 - \frac{10}{3}y + \frac{1}{9}y^2 + y^2 = 25 \quad \text{or} \quad \frac{10}{9}y(y-3) = 0.$$

So we can conclude that either $y = 0$ or $y = 3$. But since your friend has gone a couple of miles we cannot have $y = 0$. Therefore we must have $y = 3$ and so $x = 4$. So we conclude that we have traveled four miles in our canoe and our friend has traveled three miles in their canoe.

3. (a) For a tangent line to the equation $y = f(x)$ at $x = a$ express the y-intercept of the tangent line in terms of a, $f(a)$ and $f'(a)$. (Hint: rewrite the tangent line as $y = mx + b$, we are looking for the b.)

The tangent line to $f(x)$ at $x = a$ is given by

$$y = f(a) + f'(a)(x - a) = f'(a)x + \underbrace{\boxed{f(a) - af'(a)}}_{=y\text{-intercept}}.$$

(b) For $y = \dfrac{1}{1 + x^2}$, find the value $a > 0$, where the y-intercept of the tangent line at $x = a$ is maximal. (You do not need to prove that it is maximal.)

Here our function is $f(x) = \dfrac{1}{1 + x^2}$ which has a derivative of $f'(x) = \dfrac{-2x}{(1 + x^2)^2}$. So using part (a) we have that the y intercept of the tangent line at $x = a$ will be

$$f(a) - af'(a) = \frac{1}{1 + a^2} - a\frac{-2a}{(1 + a^2)^2} = \frac{1 + a^2}{(1 + a^2)^2} + \frac{2a^2}{(1 + a^2)^2} = \frac{1 + 3a^2}{(1 + a^2)^2}.$$

Therefore, we need to maximize $\dfrac{1 + 3a^2}{(1 + a^2)^2}$ for $a > 0$. To do this we need to take the derivative which we can do by using the quotient rule (or if we have forgotten the quotient rule we can do a combination of the chain rule and the product rule). And so we have the following.

$$\begin{aligned}
\frac{d}{da}\left(\frac{1 + 3a^2}{(1 + a^2)^2}\right) &= \frac{(6a)(1 + a^2)^2 - (1 + 3a^2)\big(2(1 + a^2)(2a)\big)}{\big((1 + a^2)^2\big)^2} \\
&= \frac{(6a)(1 + a^2) - (1 + 3a^2)(4a)}{(1 + a^2)^3} \\
&= \frac{6a + 6a^3 - 4a - 12a^3}{(1 + a^2)^3} \\
&= \frac{2a - 6a^3}{(1 + a^2)^3} = \frac{2a(1 - 3a^2)}{(1 + a^2)^3}
\end{aligned}$$

Since the denominator is always positive this is never undefined. The zeroes in the numerator are at $a = 0$ and $a = \pm\sqrt{3}/3$ but since we want $a > 0$ that means we only have the one critical point which is at $a = \sqrt{3}/3$, so that must be where the maximum occurs.

(It is easy to see that for $a < \sqrt{3}/3$ the derivative is positive and for $a > \sqrt{3}/3$ the derivative is negative so that this is indeed where the maximum occurs. On a side note you might notice $a = \sqrt{3}/3$ is also an inflection point of the curve. This is because $\dfrac{d}{da}\big(f(a) - af'(a)\big) = f'(a) - \big(f'(a) + af''(a)\big) = -af''(a)$. Therefore the maximum/minimum y-intercept on a "sufficiently smooth" function, if it exists, will always occur either at $a = 0$ or at an inflection point.)

4. (a) Find the tangent line to the implicitly defined curve $y^3 + 3x^2y + x^3 = 5$ at $(1, 1)$.

We already have the point. The only thing that we need is the slope, and for this we will use implicit differentiation. So taking the derivative of both sides with respect to y we have

$$\begin{aligned} 3y^2\frac{dy}{dx} + 6xy + 3x^2\frac{dy}{dx} + 3x^2 &= 0 \qquad \text{or} \\ \left(y^2 + x^2\right)\frac{dy}{dx} &= -2xy - x^2 \qquad \text{or} \\ \frac{dy}{dx} &= \frac{-2xy - x^2}{y^2 + x^2}. \end{aligned}$$

So at the point $(1, 1)$ we have

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{-2-1}{1+1} = -\frac{3}{2}.$$

So our desired tangent line is

$$y = 1 - \frac{3}{2}(x-1) \quad \text{or} \quad y = -\frac{3}{2}x + \frac{5}{2}.$$

(b) Use part (a) to estimate the y value on the implicitly defined curve if $x = \frac{11}{10}$.

The tangent line makes a good approximation to the curve (at least locally) and since $x = \frac{11}{10}$ is close to $x = 1$ we can plug $x = \frac{11}{10}$ into the tangent line found in part (a) to get a good guess for what the y-value should be. So we have

$$y \approx 1 - \frac{3}{2}\left(\frac{11}{10} - 1\right) = 1 - \frac{3}{2}\cdot\frac{1}{10} = 1 - \frac{3}{20} = \frac{17}{20} = 0.85.$$

By way of comparison the actual value is $y = 0.8447\ldots$ which is not far off.

(Note: this general idea is not far off from the method of plotting approximate solutions to differential equations, i.e., taking a value that we know and using tangent lines to move a little further and then repeating the process.)

5. (a) Find $\displaystyle\int \frac{\sin^2\theta}{1-\cos\theta}\,d\theta$.

Since this is not in the form of one of the antiderivatives that we know, let us first "massage" it until we can get something that we are more familiar with. In this case we really would like to get rid of $1-\cos\theta$ on the bottom. Since $1-\cos^2\theta = \sin^2\theta$ then let us try multiplying by the conjugate and simplifying the result. So we have

$$\begin{aligned}
\int \frac{\sin^2\theta}{1-\cos\theta}\,d\theta &= \int \frac{\sin^2\theta}{1-\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}\,d\theta \\
&= \int \frac{\sin^2\theta(1+\cos\theta)}{1-\cos^2\theta}\,d\theta \\
&= \int \frac{\sin^2\theta(1+\cos\theta)}{\sin^2\theta}\,d\theta \\
&= \int (1+\cos\theta)\,d\theta \\
&= \theta+\sin\theta+C.
\end{aligned}$$

(b) Find $\displaystyle\int \sqrt{1+\sqrt{x}}\,dx$.

This is not in the form of something that we can integrate (at least not yet!). Looking at this we see a $1+\sqrt{x}$ inside of a $\sqrt{\ \cdot\ }$, i.e., a function in a function suggesting we should use the chain rule and try substituting u for the inside term.

$$\text{If } u = 1+\sqrt{x} = 1+x^{1/2} \text{ then } du = \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}\,dx \text{ or } dx = 2\sqrt{x}\,du.$$

Now we wanted to get rid of all of the x's and replace them by u's. Looking back at the definition of u we see that we can rearrange it and solve for $\sqrt{x}$, namely we have $\sqrt{x} = u-1$. And so we have that $dx = 2(u-1)\,du$. Therefore, we have

$$\begin{aligned}
\int \sqrt{1+\sqrt{x}}\,dx &= \int \sqrt{u}\cdot 2(u-1)\,du \\
&= \int \left(2u^{3/2}-2u^{1/2}\right)du \\
&= 2\cdot\frac{2}{5}u^{5/2} - 2\cdot\frac{2}{3}u^{3/2}+C \\
&= \frac{4}{5}\left(1+\sqrt{x}\right)^{5/2} - \frac{4}{3}\left(1+\sqrt{x}\right)^{3/2}+C.
\end{aligned}$$

6. Let $h(x) = \displaystyle\int_{3x^2-2}^{2x^2+x} \frac{1}{2+\sin t}\,dt.$

(a) For what two values does $h(x) = 0$?

Since $h(x)$ is defined in terms of an integral, and we want $h(x) = 0$, then we recall that one trivial way for an integral to be 0 is if we have $\displaystyle\int_a^a f(x)\,dx$, i.e., the same upper and lower bounds. In this case we will have an integral of 0 if

$$2x^2 + x = 3x^2 - 2 \text{ or } x^2 - x - 2 = 0 \text{ or } (x-2)(x+1) = 0.$$

In particular, this will clearly happen when either $x = 2$ or $x = -1$, our two desired values.

(Note: while not strictly required by the problem it is easy to see that these are the only zeroes of $h(x)$ since the function $1/(2+\sin t) \geq 1/3$ and so if the two bounds do not match the integral must be nonzero.)

(b) Show that between the two values found in part (a) that there is a critical point. (You do not have to find the critical point. Make sure to state any theorems that you are using.)

Since we have $h(-1) = h(2) = 0$ then by the Mean Value Theorem we have that for some $-1 < c < 2$ that

$$h'(c) = \frac{h(2) - h(-1)}{2-(-1)} = 0,$$

in particular that c will be a critical point. We could also have used Rolle's Theorem to get the conclusion since $h(2) = h(-1)$ and so there must be some point in between where the derivative is 0.

(Note: technically we also need to check that the function is continuous and differentiable. But since it has a derivative, as we will see in part (c) then this condition is also easily satisfied.)

(c) Find $h'(x)$.

We use the Fundamental Theorem of Calculus together with the chain rule to get

$$h'(x) = \frac{d}{dx}\left(\int_{3x^2-2}^{2x^2+x} \frac{1}{2+\sin t}\,dt\right)$$
$$= \frac{1}{2+\sin(2x^2+x)}\cdot(4x+1) - \frac{1}{2+\sin(3x^2-2)}\cdot(6x).$$

7. Suppose that $f(x)$ is a function such that

- the average of $f(x)$ for $0 \leq x \leq 1$ is 2;
- the average of $f(x)$ for $0 \leq x \leq 4$ is 5;
- the average of $f(x)$ for $1 \leq x \leq 5$ is 3;
- the average of $f(x)$ for $2 \leq x \leq 5$ is -1.

Find the average of $f(x)$ for $2 \leq x \leq 4$.

First let us translate these statements about averages to statements about the integrals. In particular we have

$$\int_0^1 f(x)\,dx = 2, \quad \int_0^4 f(x)\,dx = 20, \quad \int_1^5 f(x)\,dx = 12 \text{ and } \int_2^5 f(x)\,dx = -3.$$

Now that we know these integrals we can piece them back together to figure out the integral from 2 to 4. One way to do this is to note that

$$\int_0^5 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^5 f(x)\,dx = 2 + 12 = 14,$$

$$\int_0^2 f(x)\,dx = \int_0^5 f(x)\,dx - \int_2^5 f(x)\,dx = 14 - (-3) = 17,$$

and so

$$\int_2^4 f(x)\,dx = \int_0^4 f(x)\,dx - \int_0^2 f(x)\,dx = 20 - 17 = 3.$$

So we have that the average of $f(x)$ for $2 \leq x \leq 4$ is

$$\frac{1}{4-2}\int_2^4 f(x)\,dx = \frac{1}{2}\cdot 3 = \frac{3}{2}.$$

8. After retiring from your job as an executive at the widget company you have decided to write a book, *The Widget Jones Diary*, which chronicles your rise from the factory floor making widgets to the leader of the largest widget manufacturer in the greater Mississauga area.

Your publisher has asked you how many copies of the book they should print, any books that do not sell after a year you have to buy and so you only want to publish as many as will sell. After some careful study of the book market you have determined that you will sell $4000/(t+8)^2$ books per month t months after the book is published.

How many books should you publish to fulfill the demand but also have no books leftover at the end of one year, i.e., you sell all books from the date of publication ($t = 0$) to one year later ($t = 12$)? (Use integration to find the answer.)

Integration is geared to finding the totals. In this case to find the total number of books that we need we need to find

$$\int_0^{12} \frac{4000}{(t+8)^2}\,dt,$$

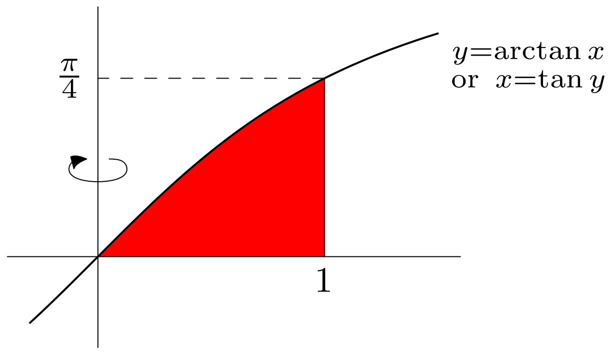
i.e., the number of books that we will between $t = 0$ (when the book comes out) and $t = 12$ one year later. So we have

$$\begin{aligned}\int_0^{12} \frac{4000}{(t+8)^2}\,dt = 4000\int_0^{12}(t+8)^{-2}\,dt &= -4000(t+8)^{-1}\Big|_{t=0}^{t=12} \\ = -\frac{4000}{t+8}\Big|_{t=0}^{t=12} &= -\frac{4000}{12+8} - \left(-\frac{4000}{0+8}\right) \\ &= -\frac{4000}{20} + \frac{4000}{8} = -200 + 500 = 300.\end{aligned}$$

Therefore you should arrange to have 300 books published.

9. Find $2\pi \int_0^1 x \arctan x \, dx$. (Hint: interpret this integral as a volume of rotation and find a different integral for the same volume.)

This integral finds the volume generated by rotating the region under $y = \arctan x$ between $x = 0$ and $x = 1$ around the y-axis (see the following picture).



In particular, we can find the same volume where instead of integrating with respect to x we integrate with respect to y, which will be the region between $x = \tan y$ and $x = 1$ from $y = 0$ to $y = \pi/4$. And so we have

$$2\pi \int_0^1 x \arctan x \, dx = \pi \int_0^{\pi/4} \big((1)^2 - (\tan y)^2\big) \, dy = \pi \int_0^{\pi/4} \big(1 - \tan^2 y\big) \, dy.$$

Of course now we have the problem of trying to integrate $\tan^2 y$. But this is not so bad since we can use the identity $\tan^2 y = \sec^2 y - 1$. So we have

$$\pi \int_0^{\pi/4} \big(1 - \tan^2 y\big) \, dy = \pi \int_0^{\pi/4} \big(1 - (\sec^2 y - 1)\big) \, dy = \pi \int_0^{\pi/4} \big(2 - \sec^2 y\big) \, dy$$

$$= \pi\big(2y - \tan y\big)\Big|_{y=0}^{y=\pi/4} = \pi\big(2\frac{\pi}{4} - \tan\frac{\pi}{4}\big) - \pi\big(2{\cdot}0 - \tan 0\big) = \frac{1}{2}\pi^2 - \pi.$$

(Note: it is also possible to do this integral by doing integration by parts with a slight dash of integration of rational functions. You will learn more about these techniques in the next class, assuming you survived this final!)

10. In ten lines or less, please tell us three things that you have learned by taking this class. (There are **no** wrong answers, you will get full credit as long as you write anything, so please be honest!)

There are many possible correct answers for this one, including "When Professor Butler tells you that a problem is easy, he is lying, he means it is super hard!"