

Final Review

Material from the previous midterms

The final is cumulative but will not include problems about limits, curve sketching and Riemann sums. More information about what we covered can be found in the previous midterm reviews. This review will only cover the material after the second midterm (i.e., integration).

Riemann sums

If we want to approximate an area we can slice it into little strips each of which can be approximated by a rectangle; we then add up the individual rectangles. To get a better approximation we can make the slices “smaller”. This is the underlying idea of *Riemann sums*. Given a function $f(x)$ and an interval $[a, b]$ we start by partitioning $[a, b]$ up into a partition into pieces by first choosing points

$$x_0 = a < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

These are the bases of the rectangle and the “width” of the i th rectangle is $\Delta x_i = x_i - x_{i-1}$. To find the heights we choose points c_i so that $x_{i-1} \leq c_i \leq x_i$ we then have that the “height” is $f(c_i)$. So then we have that the Riemann sum (which is an approximation of the area under the curve $y = f(x)$ in the interval $[a, b]$) is

$$\sum_{k=1}^n f(c_k) \Delta x_k.$$

Here “ Σ ” is used to indicate doing a sum. To indicate the slices getting smaller we let $\max\{\Delta x_k\} \rightarrow 0$. We are interested in functions where the limit as $\|P\| \rightarrow 0$ exists, we call such functions integrable (all continuous functions are integrable) and denote the limit by

$$\lim_{\max\{\Delta x_k\} \rightarrow 0} \sum_{k=1}^n f(c_k) \Delta x_k = \int_a^b f(x) dx.$$

The \int sign is a stretched out “S” and indicates the idea that we are summing up little pieces. The “ x ” is a dummy variable and can be replaced by any other variable, the result will be the same.

$$\int_a^b f(x) dx = \int_a^b f(u) du = \int_a^b f(y) dy = \cdots$$

While our starting point is thinking of finding area, it is important to remember that the result of the integration can be positive or negative, so more appropriately it is signed area.

While Riemann sums underly the principles of integration, we will *not* test you directly on Riemann sums (including problems involving Σ).

Properties of integrals

Properties of integration follow from the definition of Riemann sums (as well as some geometric intuition).

$$\int_a^b f(x) dx = \left[\begin{array}{c} \text{area above} \\ x\text{-axis} \end{array} \right] - \left[\begin{array}{c} \text{area below} \\ x\text{-axis} \end{array} \right].$$

We can find the values of some integrals by finding the area is composed of combinations of triangles, rectangles and circles (right now this is the only way we can handle integrals of the form $\sqrt{r^2 - x^2}$).

If our upper and lower bound match then there is no “area” and so the integral is 0, i.e.,

$$\int_a^a f(x) dx = 0.$$

Changing the order of integration changes the sign, i.e.,

$$\int_a^b f(x) dx = - \int_b^a f(x) dx.$$

Integration is “linear” in the sense that we can pull constants out as well as break it up over addition, i.e.,

$$\int_a^b kf(x) dx = k \int_a^b f(x) dx \quad \text{and}$$

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$$

This for example allows us to break the problem of integrating several things added together into individual parts (this is especially convenient when we need to do one trick for integrating one part and a different trick for integrating another part).

We can break the interval we are integrating into pieces (this is convenient, for example, when we have piecewise functions), i.e.,

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

This is true for any relationship of a, b, c . We can also reverse this and combine several integrals together to make a single integral.

If $f(x) \leq g(x)$ on $[a, b]$ then

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

In particular, if $m \leq f(x) \leq M$ on $[a, b]$ then

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a).$$

Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus connects the ideas of differentiation with our new idea of integration. There are two parts to the Fundamental Theorem of Calculus.

(Part I) If f is continuous on $[a, b]$ then

$$\int_a^b f(u) du = F(b) - F(a),$$

where $F(x)$ is any anti-derivative of $f(x)$.

In particular if we want to evaluate a definite integral we can now do it in two steps. First, find an anti-derivative of the function. Second, evaluate this function at the endpoints and take the difference. This reduces the problem of integration to that of finding an anti-derivatives.

Of course finding anti-derivatives in general are not easy! In general our goal is to work on rewriting the function using algebraic manipulation, trigonometric identities, or substitution (see below) so that we can reduce the anti-derivative to something that we easily recognize.

The other part of the Fundamental Theorem of Calculus says that integration leads to anti-derivatives.

(Part II) If f is continuous on $[a, b]$ and

$$F(x) = \int_a^x f(u) du \text{ then } F'(x) = f(x).$$

By combining this part of the Fundamental Theorem of Calculus, the chain rule and properties of integrals we have the following rule:

$$\frac{d}{dx} \left(\int_{h(x)}^{g(x)} f(u) du \right) = f(g(x))g'(x) - f(h(x))h'(x).$$

We see from part II of the Fundamental theorem of calculus that the function $F(x)$ is an anti-derivative of $f(x)$. So we will let $\int f(x) dx$ (called the indefinite integral) denote the anti-derivative of $f(x)$. In general we have that

$$\int f(x) dx = C + \int_a^x f(x) dx,$$

where C is a constant (this constant will play an important role later, it is important not to forget it).

Substitution rule

Rules for derivatives become rules for integration. One of the most important rules for derivatives is the chain rule which states

$$\frac{d}{dx} (f(g(x))) = f'(g(x))g'(x).$$

By taking the anti-derivative of each side we can conclude

$$\begin{aligned} \int f'(\underbrace{g(x)}_{=u}) \underbrace{g'(x) dx}_{=du} &= \int f'(u) du = f(u) + C \\ &= f(g(x)) + C. \end{aligned}$$

This is used in many problems involving integration because it can help rewrite the integral in a simpler form. So after the substitution we might see how to proceed and then we can solve the integral and at the end *resubstitute* back to get our answer in terms of x . The largest indication that we should use substitution is to look for a function inside of a function.

With every method we have in working with solving integrals the goal is always to make it simpler. In some sense the art of integration is the art of cumulative simplification. It is possible that several substitutions might be needed. This is fine as long as you keep track of everything.

Remember that when we are substituting that we need to substitute for every occurrence of " x ", i.e., we also need to make sure we substitute for the dx term. On some integrals, in order for us to do this we might need to solve for x in terms of u . For example when making the substitution $u = \sqrt{x}$ then $du = \frac{1}{2}x^{-1/2} dx$ or $dx = 2\sqrt{x} du = 2u du$ so that the appropriate substitution in this case is to replace the dx term by $2u du$. (Note it is easy to add and divide by constants to get what we need.)

If we are dealing with a definite integral we can do one of two things. First, we do the indefinite integral, solve it to the end to get an antiderivative and then use the fundamental theorem of calculus to evaluate and get our answer. Alternatively, we can change the bounds as we make our substitution (the principle is again that we are replacing *every* occurrence of x , and the original bounds were in terms of x), i.e.,

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du, \text{ where } u = g(x).$$

Applications of integration

Cumulative change: There is a connection between integration and derivatives and we can use this to answer questions about given how fast something is changing, what is the total. (These types of questions are easy to identify since they will involve only one rate (where related rates involves more than one) and will ask for a total.) By the fundamental theorem of calculus we have

$$\int_a^b f'(t) dt = f(b) - f(a) \text{ or } f(b) = f(a) + \int_a^b f'(t) dt.$$

The intuition is that $f'(t) dt$ measures the (instantaneous) amount of change at time t and then the " \int " adds them all up to find the total amount of change.

If $v(t) = s'(t)$ is velocity then

$$\int_a^b v(t) dt = s(b) - s(a) = \text{displacement},$$

and

$$\int_a^b |v(t)| dt = \text{total distance},$$

Area: We can use integration to find the area between curves. If $g(x) \leq f(x)$ on the interval $[a, b]$ then the area between these curves in the interval is

$$\text{Area} = \int_a^b (f(x) - g(x)) dx.$$

If the curves cross then find the intersection point(s) by setting $f(x) = g(x)$ and solving for x (also done when no bounds are given, or when there are several curves that define the region). Once we have the intersection point(s) we split this into several pieces and work on each piece separately (this is to avoid the problem of “signed” areas and to simplify the integrals to manageable functions).

We can also integrate with respect to y , the basic idea being to take horizontal slices. In this case if we have $x = f(y)$ and $x = g(y)$ with $g(y) \leq f(y)$ on the interval $[a, b]$. Then we have

$$\text{Area} = \int_a^b (f(y) - g(y)) dy.$$

Average value: The average value of f in the interval $[a, b]$ is given by

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x) dx.$$

This value is such that the rectangle with height f_{avg} and width $(b-a)$ has the same area as $\int_a^b f(x) dx$.

The *Mean Value Theorem for Integrals* states that if f is continuous on $[a, b]$ then there is some $c \in [a, b]$ so that

$$f(c) = f_{avg} = \frac{1}{b-a} \int_a^b f(x) dx.$$

(This is actually the same as the Mean Value Theorem for Derivatives, just worded differently.)

Volume: We can use integration to find volume. Intuitively this is done by slicing into cross sections and then adding up the cross sections. So we have that

$$\text{Volume} = \int_a^b A(x) dx,$$

where $A(x)$ is the area of the slice of the cross section.

One special case is when we form a solid by revolving a region around the axis. If we revolve the region between $y = f(x)$ and the x -axis around the x -axis then a cross section is a circle and so the area of the cross section is $\pi(f(x))^2$. So in this case we have

$$\text{Volume} = \pi \int_a^b (f(x))^2 dx.$$

If we want to spin a region between two curves $g(x)$ and $f(x)$ with $g(x) \leq f(x)$ then the resulting cross sections are annuli (or washers) and the area of the

cross sections are the difference of circles. Applying the same rule we have

$$\text{Volume} = \pi \int_a^b ((f(x))^2 - (g(x))^2) dx.$$

If we revolve the region between $y = f(x)$ and the x -axis around the y -axis then we can form cross sections that look like shells (the edge of cylinders), where the area of the cross section is $2\pi x f(x)$ (where $2\pi x$ is the diameter of the shell and $f(x)$ is the height). So in this case we have

$$\text{Volume} = 2\pi \int_a^b x f(x) dx.$$

If we want to spin a region between two curves $g(x)$ and $f(x)$ with $g(x) \leq f(x)$ then the resulting cross sections are still the same type of shells, now only the height has changed to $f(x) - g(x)$. Applying the same rule we have

$$\text{Volume} = 2\pi \int_a^b x(f(x) - g(x)) dx.$$

We can also find these same volumes by integrating with respect to y . Note that when revolving around the x -axis, if we integrate with respect to x we use the washer method and if we integrate with respect to y we use the shell method. When revolving around the y -axis, if we integrate with respect to x we use the shell method and if we integrate with respect to y we use the washer method. (Occasionally it is the case that one method is easier to integrate than the other; so it is useful to switch the method of calculating volumes between these two possibilities.)