

Parting thoughts (from Piet Hein)

The road to wisdom?

Well it's plain and simple to express:

err and err and err again
but less and less and less.

Problems worthy of attack
prove their worth by fighting back.

Math 31A Review

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About the final

- Thursday March 18, 8am-11am, Haines 39
- No notes, no book, no calculator
- Ten questions
 - \approx Four review questions (Chapters 3,4)
 - \approx Five new questions (Chapters 5,6)
 - \approx One essay question
- No limits, curve sketching, Riemann sums, or density
- Office hours:
 - Monday 8:30am - 1:30pm
 - Wednesday 10:30am - 3:30pm

Start studying well in advance. If you haven't started yet, start today!

The best way to remember **anything** is through repetition. One of the best ways to help remember things is through regular short review sessions; as compared to one last minute cram session¹.

Study what ideas and techniques we have gone over this quarter. It is one thing to be able to know an identity or a procedure; it is an entirely different thing to be able to use that identity at the **right** time and in the **right** way.

¹Also make sure to get plenty of sleep the night before the final. It will help you to focus on the arithmetic and keep you from falling asleep during the final!

Learn from your mistakes, otherwise you will repeat them!

Look back over your midterms, particularly those where you were not able to complete the problem, and ask “where did I get stuck and how can I avoid getting stuck like that again?”

It is helpful to look at the solutions and ask “how would I have thought of that?” or “what about the problem would have told me to do such and such a technique?”

**Fool me once, shame on you.
Fool me twice, shame on me.**

Have a plan to answer your question

Before starting your problem read the problem **carefully** and think about how you will answer your questions.

Look for clues about which techniques you will be using (“rate”, “-est”, “estimate” or “approximate”). Some problems might involve several different techniques.

- What am I given?
- What am I trying to find?
- How are these connected?

Failure to plan is planning for failure.

If something looks impossible to do, maybe we don't have to do it!²

Some problem might seem to have something which is difficult or even impossible to do using what we have learned. But perhaps the problem is testing us on something else, take a moment and think “do I **really** have to do this?”

Or perhaps “maybe this is one of professor Butler’s misdirection problems, what are we **really** being tested on?” (It might not be what you think you are being tested on!)

²Unfortunately, sometimes the answer is “yes, you do have to do it”

Work to maximize your score

Make sure you do the problems that you **can** do before you spend your time stuck on problems you **cannot** do.

Even if you cannot do a whole problem **do as much as possible** of the problem as you can (i.e., a problem might be testing several different things; try to get as far as possible in each avenue before stopping).

Check your work (continually)!³ Check your arithmetic and algebra! If you are unsure whether something is legal or not, it probably is not! Leave yourself time to go back over your answer and change it if needed (be careful not to change a right answer to a wrong answer).

³It helps if you write in a clean and well organized manner

Don't worry; be happy!

Relax! Don't go into the final with a mindset of "I have to get a perfect score" or "I can only miss two problems". Remember that *no one* (including the professor) can say for sure what grade you need on the final to reach a certain point; your job is to do your best on the whole exam.

Remember, no matter what grade you get, Professor Butler still likes you and you will still have a great life!

Rates of change

The first part of calculus looks at how **fast** something is changing, i.e., rates of change.

Average rate of change from $x = a$ to $x = b$ corresponds to slope of secant line through these points,

$$\frac{f(b) - f(a)}{b - a}.$$

Instantaneous rate of change at $x = c$ corresponds to slope of tangent line of function at the point, $f'(c)$.

The derivative

The idea behind the derivative is to find the slope of the tangent line which is done by taking secant lines and doing better and better approximations,

Derivative

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{a \rightarrow x} \frac{f(a) - f(x)}{a - x}$$

While derivatives theoretically **can** be found using this limit definition, in practice it is much easier to apply the rules for taking derivatives of basic functions together with the linearity, product, quotient and chain rules.

(For piecewise functions take the derivative of each piece using the rules and see what happens at the points they glue together.)

$$\frac{d}{dx}(x^a) = ax^{a-1}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(k \cdot f(x)) = k \cdot f'(x)$$

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

Tangent lines

With the slope and the point we can get the tangent line.

Tangent line to $y = f(x)$ at $x = a$

$$y - f(a) = f'(a)(x - a) \text{ or } y = f(a) + f'(a)(x - a).$$

Anytime you can take a derivative you can also find a tangent line, so tangent lines can show up almost anywhere (including derivatives of integrals).

Things to watch out for

- Remember to evaluate $f(a)$ and $f'(a)$ before putting them into the tangent line.
- Sometimes we might be looking for tangent lines satisfying certain properties (i.e., certain slopes).

Implicit differentiation

An implicit function is a relationship between y and x where we cannot explicitly solve for y . Even though we do not have $y = f(x)$, we can still find y' in terms of y and x .

- Take the derivative of both sides with respect to x .
- When taking the derivative of y write y' (or $\frac{dy}{dx}$).
- Rearrange and solve for y' .

Things to watch out for

- Sometimes it is better to simplify an expression before taking a derivative (i.e., clear denominators).
- It is helpful to think of y as $y(x)$, a function of x and to use the chain rule appropriately.

Related rates

If two quantities are related then their rates of change (with respect to time) are also related.

- Look for a relationship between the quantities that are changing. (Pythagorean theorem, similar triangles, areas, volumes, etc.)
- Take derivative of both sides with respect to t .
- Solve for the desired quantity (it does **not** have to be a rate).

Things to watch out for

Make sure your answer is reasonable!

Linear approximation

The tangent line tells us how the function is behaving **near** the point of tangency. So we can use the tangent line to approximate what is happening with the function.

$$\Delta f \approx f'(a)\Delta x$$

- Use when we want to estimate the **difference** in some quantity.

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

- Use when we want to estimate the **value** of some quantity.

Finding extrema

Extrema (i.e., max/min) are found by looking for where they can occur (i.e., critical points, $f' = 0$ or $f' = \text{DNE}$, and boundary points) and then classifying what they can be.

First derivative test

Look at how the first derivative tells you behavior about increasing/decreasing to identify whether a point is a minimum, a maximum or neither. $\nearrow \searrow = \text{MAX}$, $\searrow \nearrow = \text{MIN}$, $\nearrow \nearrow = \searrow \searrow = \text{NEITHER}$.

Things to watch out for

- Ignore critical points that are outside our domain (i.e., for example when finding global maximum).
- Choose “easy” point to determining the sign of the first derivative on an interval.

Second derivative and concavity

The second derivative tells us how the curve is “bending”; $f'' > 0$ the curve is bending up or concave up; $f'' < 0$ the curve is bending down or concave down.

Second derivative test

If $f'(a) = 0$ and $f''(a) > 0$ then point is a minimum, if $f'(a) = 0$ and $f''(a) < 0$ then point is a maximum, in all other cases the second derivative test is inconclusive.

An inflection point is a point where concavity of the curve changes from up to down or from down to up. (Essentially the critical points of the first derivative.)



Rolle's Theorem and Mean Value Theorem

Rolle's Theorem

If $f(x)$ is continuous on $a \leq x \leq b$, differentiable on $a < x < b$ and $f(a) = f(b)$, then there is some $a < c < b$ so that $f'(c) = 0$ (i.e., a critical point between a and b).

Mean Value Theorem

If $f(x)$ is continuous on $a \leq x \leq b$ and differentiable on $a < x < b$, then there is some $a < c < b$ so that

$f'(c) = \frac{f(b) - f(a)}{b - a}$, i.e., at some time we are exactly average.

We could be asked to find a point(s) satisfying the theorem, but we can also use these theorems to show that a point with some property exists.

Optimization

When looking for things which are “-est” we use derivatives.

- Express what we are optimizing as a function of a **single** variable (might need to use constraints to do this).
- Take derivatives and find critical points; then apply test to determine which critical point(s) give the desired answer.

Things to watch out for

- Make sure your answer is reasonable.
- Carefully check to see what we are asked to find and give your answer in the appropriate form, i.e., an x value, an area, a volume.

Antiderivatives

We say $F(x)$ is the antiderivative of $f(x)$ if $F'(x) = f(x)$.

Antiderivatives are not unique, they can differ by a constant.

$$\int f(x) = F(x) + C$$

$$\int x^a dx = \frac{1}{a+1} x^{a+1} + C$$

$$\int \frac{1}{1+x^2} dx = \arctan x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

Integration

The idea behind integration is to slice what we are interested in finding (i.e., area) into little pieces that we can manage. We then add up all of the little pieces and get the result. If we do this for the function $f(x)$ where we slice between $x = a$ and $x = b$ we get the following interpretation.

$$\int_a^b f(x) dx = \left[\begin{array}{c} \text{area above} \\ x\text{-axis} \end{array} \right] - \left[\begin{array}{c} \text{area below} \\ x\text{-axis} \end{array} \right].$$

Things to watch out for

In cases where our area is made of triangles, squares and/or circles it is usually better to find the integral by finding the appropriate area.

Properties of integrals

- $\int_a^a f(x) dx = 0.$
- $\int_a^b f(x) dx = - \int_b^a f(x) dx.$
- $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$
- $\int_a^b k \cdot f(x) dx = k \int_a^b f(x) dx.$
- $\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$

Comparing integrals

If $f(x) \geq g(x)$ for $a \leq x \leq b$ then

$$\int_a^b f(x) \geq \int_a^b g(x) dx.$$

If $m \leq f(x) \leq M$ on the interval $a \leq x \leq b$ then

$$m \leq \frac{1}{b-a} \int_a^b f(x) dx \leq M.$$

Things to watch out for

We only care about what is happening between $x = a$ and $x = b$, i.e., it is not important what the functions do **outside** this range.

Differentiation and integration are connected

Fundamental Theorem of Calculus I

If f is continuous for $a < x < b$ then

$$\int_a^b f(x) dx = F(b) - F(a),$$

where $F(x)$ is **any** anti-derivative of $f(x)$.

Fundamental Theorem of Calculus II

If f is continuous then

$$\frac{d}{dx} \left(\int_a^x f(t) dt \right) = f(x).$$

Application of FTC I: Easier integrals

If we know an antiderivative of $f(x)$ we can easily integrate. Functions that we know antiderivatives for include:

$$x^a \ (a \neq -1), \quad \frac{1}{1+x^2}, \quad \sin x, \quad \cos x, \quad \sec^2 x, \quad \sec x \tan x$$

Things to try when looking for antiderivatives

- Expanding polynomials.
- Multiplying by the conjugates.
- Using trigonometric identities.
- Breaking into a sum of simpler pieces.

Application of FTC II: Derivatives of integrals

Combining the fundamental theorem of calculus with the chain rule we have the following.

$$\frac{d}{dx} \left(\int_{g(x)}^{h(x)} f(t) dt \right) = f(h(x))h'(x) - f(g(x))g'(x).$$

Another application is that this shows **every** continuous function has an antiderivative, i.e., $F(x) = \int_a^x f(t) dt$. To find a formula for the antiderivative it often helps to break it up into a piecewise function.

Substitution rule

Rules for integration come from rules for differentiation. The most important rule for integration is the **substitution** rule which comes from the **chain rule**.

$$\int f(g(x))g'(x) dx = \int f(u) du \quad \text{where} \quad \begin{array}{l} u = g(x) \\ du = g'(x) dx \end{array}$$

For definite integrals we have

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

Things to watch out for

- Look for functions inside of functions.
- Might need to solve for x in terms of u , i.e., rewrite $dx = (\text{stuff with } u) du$.

Applications of integration: Cumulative total

Given a **rate** at which a function is changing we can find the **total** amount of change, i.e.,

$$\int_a^b f'(t) dt = f(b) - f(a) \quad \text{or} \quad f(b) = f(a) + \int_a^b f'(t) dt.$$

Things to watch out for

While this problem involves rates, it is easy to distinguish this from a related rates problem. Namely in this kind of problem we will be given a rate and we will want to find a total (or something related to a total).

Applications of integration: Area

When finding the area between curves $y = f(x)$ and $y = g(x)$, if $f(x) \geq g(x)$ between $x = a$ and $x = b$ then the area is given by

$$\text{Area} = \int_a^b (f(x) - g(x)) dx.$$

Things to watch out for

- If no bounds are given then we need to solve for intersection, i.e., $f(x) = g(x)$.
- If the curves intersect or several curves bound the region then break the problem into smaller pieces.
- Sometimes it is easier to do the integral with respect to y .

Applications of integration: Average value

The average value, f_{avg} of the function on the interval $a \leq x \leq b$ is the value so that the rectangle with width $(b - a)$ and height f_{avg} has the same “area” as under the curve, i.e.,

$$\text{Average value} = f_{avg} = \frac{1}{b - a} \int_a^b f(x) dx$$

Things to watch out for

- Average value can be negative (it can even be 0).
- Mean value theorem tells us that for a continuous function on $a \leq x \leq b$ there is some c between a and b so that $f(c) = f_{avg}$.

Applications of integration: Volume I

We find volume by adding up the area of cross sections. When we form a solid by spinning a region around the x -axis then the cross sections are **washers** and so finding volume is easy! Namely, if $f(x) \geq g(x) \geq 0$ and we revolve the region between these curves and $x = a$ and $x = b$ around the x -axis, the resulting volume is

$$\text{Volume} = \pi \int_a^b ((f(x))^2 - (g(x))^2) dx.$$

Things to watch out for

- Don't forget to square the functions and be careful when simplifying.
- Volume (as with area) should never be negative.

Applications of integration: Volume II

We can also spin regions around the y -axis. In this case instead of breaking volume into small washers we will break the volume up into small **shells**. So if $f(x) \geq g(x) \geq 0$ and we revolve the region between these curves and $x = a$ and $x = b$ around the y -axis, the resulting volume is

$$\text{Volume} = 2\pi \int_a^b x(f(x) - g(x)) dx.$$

Things to watch out for

By changing what variable we integrate with respect to we can express a given volume using two different methods, i.e., using the shell and washer methods. Sometimes one of the two integrals is much easier to do.