MATH 3B (Butler)
Practice for Final (II)

Try to answer the following questions without the use of book, notes or calculator. Time yourself and try to finish the test in less than 3 hours.

1. Let $G(x) = \int_{\sin x}^{x^2} e^{t^2}\,dt$. Find the degree 2 Taylor polynomial about $x = 0$ for $G(x)$.

2. For y between 0 and 4 find the area between the curves $g(y) = 4y^2 + 8y + 23$ and $h(y) = -2y^2 + 26y + 23$.

3. Find $\int_{1/2}^{1} r(4x)\,dx$ given

$$\int_1^3 r(2x)\,dx = 5, \qquad \int_1^2 r(3x)\,dx = 4, \qquad \int_4^3 r(x)\,dx = 2 \quad \text{and} \quad \int_0^1 r(5x)\,dx = 6.$$

4. Find $\int_0^\infty \frac{\sqrt{y}}{y^3+1}\,dy$.

(Hint: you do **not** need to know that $y^3 + 1 = (y+1)(y^2 - y + 1)$.)

5. (a) Use substition to show that for a function $f(x)$ that

$$\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx.$$

(b) Show that

$$\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \frac{1}{2}a.$$

(Hint: $\int_0^a g(x)\,dx = \frac{1}{2}\big(\int_0^a g(x)\,dx + \int_0^a g(x)\,dx\big)$, and use part (a).)

(c) Find $\int_0^{\pi/2} \frac{1}{1+\tan x}\,dx$. (Hint: $\sin x = \cos\left(\frac{\pi}{2} - x\right)$.)

6. Find $\int \frac{e^x - 1}{e^x + 1}\,dx$.

7. Consider the autonomous differential equation

$$\frac{dy}{dt} = \frac{1-y^2}{1+y^2}.$$

(a) Find the equilibrium solutions to the differential equation, and determine the stability of each equilibrium point.

(b) If the initial condition $y(0) = 2$ is given, what happens to y as $t \to \infty$? (Hint: you do not need to solve the differential equation to answer this part.)

8. (a) Find an equation for the line in parametric form which passes through the point $(2, -1, 4)$ and is perpendicular to the plane $x - 3y + 4z = -5$.

(b) Find the point in the plane $x-3y+4z = -5$ closest to the point $(2, -1, 4)$. (Hint: this point *must* lie on the line found in part (a), so we need to find the right "t" so that the point in the line is also in the plane.)

9. Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x + x^2y^2}$. (Hint: break the function into two pieces.)

Note: technically we should also define what the function should be for $x = 0$, ignore this technicality.

10. One way to describe a surface is as $z = f(x, y)$, but we can also describe a surface by $F(x, y, z) = c$, where c is a fixed constant. The tangent plane of $F(x, y, z) = c$ at the point (x_0, y_0, z_0) is given by

$$\frac{\partial F(x_0, y_0, z_0)}{\partial x}(x - x_0) + \frac{\partial F(x_0, y_0, z_0)}{\partial y}(y - y_0) + \frac{\partial F(x_0, y_0, z_0)}{\partial z}(z - z_0) = 0$$

Find the tangent plane to the ellipsoid (the three dimensional analog of an ellipse) $x^2 + 4y^2 + 9z^2 = 34$ at the point $(3, -2, -1)$.

11. Find the gradient for $g(x, y) = 3x^2y - \sin(xy) + e^y - \ln(x^2 + 1)$.

12. You have been hired to design a new shipping container. The design is simple enough, a rectangular box, so your main job is to decide what dimension to make the box. The materials to make the top/bottom sides cost $\$2/m^2$ the materials to make the front/back sides cost $\$3/m^2$ and the material to make the left/right sides cost $\$5/m^2$. If the container must holds $240\,m^3$, find the dimensions of the box that minimize the cost.