MATH 3B (Butler)
Practice for Final (I)

Try to answer the following questions without the use of book, notes or calculator. Time yourself and try to finish the test in less than 3 hours.

1. Gabriel's horn is a mathematical object taken by rotating the curve $y = \frac{1}{x}$ around the x-axis for $1 \leq x < \infty$. It can be shown (though not by what we have done) that the resulting *surface* of the horn has infinite area, i.e., it is impossible to paint the outside of the horn. Find the volume of Gabriel's horn (i.e., find how much paint it would take to fill the inside of the horn).

2. Reduce the following to a single integral of the form $A\int_B^C f(x)\,dx$ for some constants A, B, C.

$$\int_0^5 f(x)\,dx - \int_3^3 f(e^x)\,dx + \int_0^1 3f(3x)\,dx - \int_0^4 f(\frac{1}{2}x)\,dx + \int_5^3 f(x)\,dx.$$

3. Find the average value for $f(x) = |x^2 - 4|$ for the interval $-3 \leq x \leq 5$.

4. Given $g(0) = 3$, $g'(0) = -1$, $g''(0) = \pi$, $g(1) = 5$, $g'(1) = 2$ and $g''(1) = 1$, find

$$\int_0^1 x^2 g'''(x)\,dx.$$

5. Find $\displaystyle\int \frac{2\ln(x+1)}{x^3}\,dx$.

6. Scientists have recently discovered a new species of tree, *gumdropus delectus*, which aside from its unusually sticky fruit has also been shown to exhibit some unusual growth pattern. Namely, the height H of the tree (measured in feet) satisfies the differential equation

$$\frac{dH}{dt} = \frac{1}{100}(40 - H)^2,$$

where t is time measured in years.

(a) What is the height of the tree five years after a seed is planted?

(b) What is the maximum height that the tree can grow to (i.e., what is the limit as $t \to \infty$ for the function H)?

7. Find $\int_0^{\pi/2} (\sin x + \cos x)^2 \sin x \, dx$.

8. Taylor series can be defined for multivariable functions. For instance for a function of two variables x and y the degree 2 Taylor polynomial approximation around the point $(0,0)$ is given by

$$P_2(x,y) = f(0,0) + \frac{\partial f(0,0)}{\partial x} x + \frac{\partial f(0,0)}{\partial y} y + \frac{1}{2}\frac{\partial^2 f(0,0)}{\partial x^2} x^2 + \frac{\partial^2 f(0,0)}{\partial x \partial y} xy + \frac{1}{2}\frac{\partial^2 f(0,0)}{\partial y^2} y^2.$$

(a) Find the degree 2 Taylor polynomial approximation for the function $f(x,y) = e^x\sqrt{y+9}$ around the point $(0,0)$.

(b) Use the answer in part (a) to find an approximation for $\frac{\sqrt{10}}{e}$

9. A ball is placed on a surface and released. If the surface is described by $z = 3x^2 + xy - 2y^2$ and the initial placement corresponds to the point above $x = 2$ and $y = 1$, find a unit vector (in the plane) which points in the direction that the ball will roll.

10. Consider the function $h(x,y) = y^2 \sin x - x$.

(a) Verify that $\nabla h(0,1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(b) Determine if the point $(0,1)$ is a maximum, minimum or saddle.

11. Find the maximum and minimum values for $\kappa(x,y) = 3x - y$ for points on the curve $x^2 + 4y^2 = 10$.